

Recitation 5

Zach Swaziek
Econ 8108
Adapted from
Prof. Amador's Notes

Krusell-Smith (1998) : Heterogeneous Agent
Incomplete Markets Model with Aggregate Shocks

In the Aiyagari model we studied in class, there were no aggregate shocks to the economy. The only shocks were idiosyncratic shocks faced by each agent (i.e. employment shocks). This fact made the model tractable and allowed us to fairly easily solve for the stationary equilibrium numerically; the distribution of assets and employment shocks $\Gamma(a, s)$ was invariant in the equilibrium and thus so were prices (R and w). Thus in the Aiyagari model, the agents did not have to track the evolution of Γ (in order to forecast prices) since it was invariant in the stationary equilibrium.

- Γ invariant $\Rightarrow K, L$ constant $\Rightarrow R, w$ constant.

This changes when we add aggregate shocks; i.e., shocks that affect the entire economy.

The aggregate shock we will add to the Aiyagari model is a productivity/technology shock, say A .

Because of these random technology shocks, the cross sectional distribution of assets and employment,

$\Gamma(a, s)$, may vary over time. Thus Γ

is a state variable when we add aggregate shocks to the Aiyagari model.

So the state variables are then: $(\underbrace{a, s}_{\text{Individual}}; \underbrace{\Gamma, z}_{\text{Aggregate}})$

With Γ as an additional state variable, numerically solving the value function is extremely

difficult to impossible. Γ can be a

very high dimensional object, making

numerical methods for solving the Ht problem essentially impossible. Curse of dimensionality!

Krusell-Smith (1998) discovered one way of tackling this issue. First, will go through the set up of their model.

KS Model Setup

- Technology : $f(K, L, A) = AK^\alpha L^{1-\alpha}$
- $A \in \{A_L, A_H\}$ is random
 - ↳ There are good (A_H) and bad (A_L) aggregate shocks
- Labor supplied inelastically : $l = \epsilon \bar{l}$ where ϵ is stochastic and $\epsilon \in \{0, 1\}$.
 - ↳ Can be employed ($\epsilon = 1$) or unemployed ($\epsilon = 0$)
- $\phi = 0$; Equal to natural borrowing limit
- Two sources of randomness :
 - i) Aggregate shock, A
 - ii) Idiosyncratic household shock ϵ ,Both follow some Markov chain.

In KS, aggregate shocks and household shocks are correlated so that a higher fraction of agents are unemployed during bad aggregate shocks (i.e. recessions)

Now, what is the state of the economy? (aggregate state)

$$\equiv (\Gamma, A)$$

$$a \begin{bmatrix} \epsilon \\ \vdots \\ \vdots \end{bmatrix} \leftarrow \begin{array}{l} \uparrow \text{Distribution} \\ \text{of asset } a \\ \text{and employment } \epsilon \end{array} \quad \begin{array}{l} \uparrow \text{Current} \\ \text{aggregate} \\ \text{productivity} \end{array}$$

Need to know the law of motion for Γ :

$$\underbrace{\Gamma'}_{\text{New Distribution}} = H(\underbrace{\Gamma, A}_{\text{Current state of economy}}, \underbrace{A'}_{\text{New aggregate shock}}), \quad \forall A' \in \{A'_L, A'_H\}$$

Note that there is a law of motion for each

A' , as employment distribution tomorrow may depend on A' (e.g. if more (less) likely to be employed if in good (bad) aggregate shock).

What are the state variables for an individual agent?

$$\equiv (\underbrace{a, \epsilon}_{\text{Individual states}}; \underbrace{\Gamma, A}_{\text{Aggregate states}})$$

From KS pg 873: "The role of the aggregate state is to allow the consumer to predict future prices."

Now we can write the household problem.

Household Problem

$$v(a, \epsilon; \Gamma, A)$$

$$= \max_{c, a'} \left\{ u(c) + \beta \sum_{A'} \sum_{\epsilon'} \pi(A'|A) \pi(\epsilon'|\epsilon, A) v(a', \epsilon'; \Gamma', A') \right\}$$

$$\text{s.t.} \quad c + a' = R(\Gamma, A)a + w(\Gamma, A)\epsilon \in \bar{\ell}$$

$$a' \geq 0$$

$$\Gamma' = H(\Gamma, A, A') \quad \forall A' \in \{A_L, A_H\}$$

Let $g(a, \epsilon; \Gamma, A)$ be the optimal savings policy.

Note that prices R and w are

functions of the state of the economy (Γ, A) .

Given the production function, the pricing functions are:

$$R(\Gamma, A) = 1 + A\alpha \left(\frac{K(\Gamma, A)}{L(\Gamma, A)} \right)^{\alpha-1} - \delta$$

$$w(\Gamma, A) = A(1-\alpha) \left(\frac{K(\Gamma, A)}{L(\Gamma, A)} \right)^{\alpha}$$

$K(\Gamma, A)$ and $L(\Gamma, A)$ are aggregate capital and labor supply, respectively.

This should provide some intuition on why individuals need to know Γ and its evolution.

To know prices tomorrow, R' and w' , the agents need to know aggregate capital (K') and labor (L') next period.

$$K'(\Gamma', A') = \sum_a \sum_{\epsilon} a \Gamma'(a, \epsilon)$$

$$L'(\Gamma', A') = \sum_a \sum_{\epsilon} \epsilon \bar{l} \Gamma'(a, \epsilon)$$

which are functions of $\Gamma' (= H(\Gamma, A, A'))$. And the agents need to know Γ to know Γ' (via H). How about prices two periods ahead?

Need K'' and N'' (conditional on a shock sequence):

$$K''(\Gamma'', A'') = \sum_a \sum_{\epsilon} a \Gamma''(a, \epsilon)$$

$$N''(\Gamma'', A'') = \sum_a \sum_{\epsilon} \epsilon \bar{l} \Gamma''(a, \epsilon)$$

$$\text{And } \Gamma'' = H(\Gamma', \tilde{A}', A'') = H(H(\Gamma, A, A'), \tilde{A}', A'')$$

Thus, conditional on knowing the sequence of aggregate shocks, the agents can perfectly forecast prices since they can perfectly forecast aggregate capital and labor!

Now we will define a RCE.

Advanced Aside

What if optimal savings policy took the form

$$g(a, \epsilon; \Gamma, A) = b(\epsilon; A) + m a$$

$\underbrace{\quad}_{\text{constant slope}}$

$$\begin{aligned}\text{Then } K' &= \mathbb{E}[g(a, \epsilon; \Gamma, A)] \\ &= \mathbb{E}[b(\epsilon; A) + m a] \\ &= \mathbb{E}[b(\epsilon; A)] + m \mathbb{E}[a] \\ &= \mathbb{E}[b(\epsilon; A)] + m K\end{aligned}$$

But if assume LLN ^{for ϵ} hold, then $\mathbb{E}[b(\epsilon; A)] \equiv \psi_A$ takes on a single value for each A .

$$\text{Thus } K' = G(A, K) \equiv \psi_A + m K$$

and we don't need to

track Γ to forecast future aggregate capital!

We could just track the evolution of aggregate capital instead of Γ in this case.

This gives some insight into KS's result that we will see later; the linearity of the asset policy function is important.

A Recursive Equilibrium is:

- i) Value function v
- ii) Policy function g
- iii) Pricing functions R, w
- iv) Law of motion H , such that

1) Given pricing functions and H , v and g solve HH problem

2) R, w are result of competitive factor markets

3) H is generated from g and the Markov chain for (A, ϵ) :

$$\begin{aligned}\Gamma'(a', \epsilon') &= H(\Gamma, A, A')(a', \epsilon') = \\ &= \sum_{\epsilon} \sum_{a: g(a, \epsilon; \Gamma, A) = a'} \pi(\epsilon' | \epsilon, A') \Gamma(a, \epsilon)\end{aligned}$$

↑ or "beliefs"
Agents' perceptions H about Γ evolution are correct.

How would can we compute this equilibrium?

Computational Strategy

Why not just use VFI or another numerical technique?

Problem: Γ is a high-dimensional object!

"It is well known that numerical solution of dynamic programming problems becomes increasingly difficult as the size of the state space increases." (pg. 874)
I.e. curse of dimensionality!

To see this, consider the case when assets can only take on 50 values and ϵ can only take on 2 values. Then there are 100 possible (a, ϵ) pairs, and $\Gamma(a, \epsilon) \in [0, 1]$ for all pairs. So Γ as a state essentially equates to 100 additional state variables!

This would be very difficult (if not impossible) to solve already; and we usually don't want to restrict an agent's ^{asset} choices to a finite set of values anyway! If asset choices were on an interval, this would lead to Γ essentially representing an infinite number of state variables!

KS deals with this issue by assuming that agents are "boundedly rational" about how they perceive the evolution of Γ .

KS assumes agents "do not perceive current or future prices as depending on anything more than the first I moments of Γ (in addition to z)"

Let $m = (m_1, \dots, m_I)$ be the first I moments of Γ .

The perceived law of motion is then:

$$m' = H_I(m, A, A')$$

Diagram annotations:

- A blue arrow points from the text "Incorred in principle" to the A, A' arguments of the function H_I .
- A blue bracket is drawn under the m argument, with the text "Replaced high dimensional object with a low(er) one" written below it.

Given H_I , we can solve the HH problem.

Now, the state variables are $(a, \epsilon; m, A)$.

Let the solution to the HH problem under the

"boundedly rational" beliefs be $g_I(a, \epsilon; m, A)$.

How can we check how good of an approximation this economy is?

Let $M(\Gamma)$ be a function that computes the first I moments of Γ . Given the "mistaken" beliefs about H , we can compute the exact evolution of Γ (call it H_T).
 $T = \text{True}$

$$H_T(\Gamma, A, A')(a', e') = \Gamma'(a', e') = \sum_{\epsilon} \sum_{a: g_{\pi}(a, \epsilon; \underbrace{M(\Gamma)}_{I \text{ moments of } \Gamma}, A) = a'} \pi(\epsilon' | \epsilon, A') \Gamma(a, \epsilon)$$

Thus we have the true law of motion H_T in this "boundedly rational" economy as well as the agent's perceived law of motion H_I .

We can then compare how well H_I approximates H_T .

When the difference between $\underbrace{H_I}_{m'}$ and $\underbrace{M(H_T)}_{\text{Moments of } H_T(\Gamma) = \Gamma'}$ is very small, we have an "approximate equilibrium;" the forecast errors made by the households tracking m moments is very small. Thus their forecasts of prices will also be very accurate.

"Our approximate equilibrium is a function H_I , that, when taken as given by agents,

(i) yields the best fit within the class

$\mathcal{I}$ to the behavior of m in the simulated data and

(ii) Yields a fit that is close to perfect in the sense that H_I tracks the behavior of m in the simulated data almost perfectly, that is, with small errors." (pg. 875)

Key Result: To get a good approximation, it suffices to track just the mean of Γ !

More specifically, just aggregate capital!

This is an incredibly useful result for computing the equilibria; you can replace a high dimensional object Γ with a single variable in the state space!

Now I will write down the exact algorithm KS used. See pages 875-876 or page 297-298 of Fatih Guvenen's "Macroeconomics with Heterogeneity."

1) Select I , the number of moments of Γ the agents will track.

↳ KS sets $I = 1$. So agents perceive that prices depend only on K and A .
mean of $r(a)$

2) Select functional form of H_I and the initial parameters of this function.

↳ KS selects H_I to be log linear in aggregate capital, K . So $H_I(K, A, A') =$

$$\log K' = a_0 + a_1 \log K \quad \text{if } A = A_H$$

$$\log K' = b_0 + b_1 \log K \quad \text{if } A = A_L$$

You should make an educated guess for

(a_0, a_1, b_0, b_1) . Or you could start with

e.g. $(0, 1, 0, 1)$ ($\Rightarrow K' = K$).

3) Solve household problem using H_I :

$$v(a, \epsilon; K, A) =$$

$$\max_{a', c} \left\{ u(c) + \beta \mathbb{E} [v(a', \epsilon'; K', A') | A, \epsilon] \right\}$$

$$\text{s.t.} \quad c + a' = R(K, A)a + w(K, A)$$

$$\log K' = a_0 + a_1 \log K \quad \text{if } A = A_H$$

$$\log K' = b_0 + b_1 \log K \quad \text{if } A = A_L$$

$$c, K' \geq 0$$

4) Use the savings policy function you solved for in step (3) to simulate asset holdings for N agents over T periods (N, T large numbers).
KS uses $N=5,000$ and $T=11,000$.

5) Calculate aggregate capital in each period as

$$\tilde{K}_t = \frac{1}{N} \sum_{n=1}^N a_{t,n}. \quad \text{Using } \{\tilde{K}_t\}_t \text{ run the}$$

$$\text{regressions } \log \tilde{K}_{t+1} = a_0 + a_1 \log \tilde{K}_t \quad \text{if } A = A_H$$

$$\log \tilde{K}_{t+1} = b_0 + b_1 \log \tilde{K}_t \quad \text{if } A = A_L$$

which provides you with new estimators ($\hat{a}_0, \hat{a}_1, \hat{b}_0, \hat{b}_1$) as well as R^2 and forecast variance (goodness of fit measures).

6) If $(\hat{a}_0, \hat{a}_1, \hat{b}_0, \hat{b}_1)$ is very close to (a_0, a_1, b_0, b_1) and R^2 is very high + forecast variance very small, you are done.

If not, keep iterating on 4 and 5.

(Note: Could also change functional form of H_I if you are not getting a good fit). ■

KS explains: "The idea is thus to find a fixed point for H_I in the form of a vector $(a_0^*, a_1^*, b_0^*, b_1^*)$. If this H_I is satisfactorily reproduced in simulations (i.e. if goodness of fit is high), stop. Otherwise, consider a more flexible functional form for H_I or add another moment."

(pg. 876). The approximate equilibrium obtained by KS is:

$$\log K' = 0.095 + 0.962 \log K ; R^2 = 0.999998, \hat{\sigma} = 0.0028\% \\ \text{if } A = A_H$$

$$\log K' = 0.085 + 0.965 \log K ; R^2 = 0.999998, \hat{\sigma} = 0.0036\% \\ \text{if } A = A_L$$

Regression Error

The R^2 is very, very high!

KS also shows that agents can forecast prices very, very accurately, even 25 years into the future (max. error $< 0.1\%$).

KS's model exhibits "approximate aggregation."

→ See last page of these notes

Figure 2: Consumption function is nearly linear with $\approx$ same slope across employment states. Remember in that in 8106, we discussed an aggregation theorem that loosely stated that aggregate demand is independent of the distribution of wealth when all agents' MPC out of wealth were equal (note that was for a static economy); the only thing that mattered for aggregate demand was total wealth. In the KS model:

- i) Rich agents have similar MPCs out of wealth and also hold most of the capital.
- ii) Even though poor agents have more non-linear MPCs, they consume a small percentage of wealth, so they don't matter much in the aggregate.

⊗ You can check Fatih Guvenen's "Macroeconomics with Heterogeneity" page 259. This is a great paper.

This leads to "almost perfect" aggregation.

Other Issues

- Wealth distribution in model not as dispersed as in data!
 - ↳ Wealth Gini coefficient is 0.25 in model vs. 0.79 in data!
- Make β stochastic to "fix" this; Gini in the stochastic- β model is 0.82!

Other References : • Jesús Fernández-Villaverde's slides "Krusell-Smith Models."

- Ljungqvist and Sargent Chapter 18

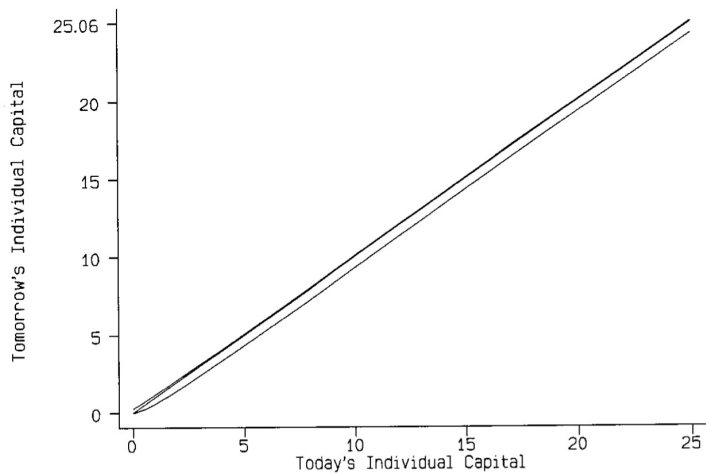


FIG. 2.—An individual agent's decision rules (benchmark model, aggregate capital = 11.7, good aggregate state).

TABLE 1
DISTRIBUTION OF WEALTH: MODELS AND DATA

MODEL	PERCENTAGE OF WEALTH HELD BY TOP					FRACTION WITH WEALTH < 0	GINI COEFFICIENT
	1%	5%	10%	20%	30%		
Benchmark model	3	11	19	35	46	0	.25
Stochastic- β model	24	55	73	88	92	11	.82
Data	30	51	64	79	88	11	.79