

## Two-Sided Lack of Commitment

- Kocherlakota (1996); follow L-S Chap 21 treatment.
- Two risk averse HHs: (A) and (B)
- Stochastic Endowments:  $\left. \begin{array}{l} \text{(A)} \ y(s^t) \\ \text{(B)} \ 1 - y(s^t) \end{array} \right\}$  No aggregate risk
- $s$  is iid ;  $\pi(s^t)$  is probability over  $s^t$
- Contract design problem:
  - ↳ Create insurance / transfer arrangement b/t (A) + (B) that:
    - i) Reduces consumption risk and also
    - ii) Respects both HH's participation constraints
      - ↳ Never optimal for either HH to exit to autarky
- Set up a planner problem (1<sup>st</sup> seq. prob, then rec. frm)
- Planner allocates consumption goods b/t HHs s.t.
  - i) Participation constraints for (A) + (B) satisfied
  - ii) Promise keeping constraint satisfied
- Planner chooses  $\{C_A\}_{s^t} \equiv \{\{C_t^A(s^t)\}_{s^t}\}_{s^t}$ 
  - ↳ Note :  $C_t^B(s^t) = 1 - C_t^A(s^t)$

•  $\{c^A\}_{s^t}$  is

i) Feasible if  $c_t^A(s^t) \in [0, 1] \quad \forall t, s^t$

ii) Sustainable if it satisfies  $\forall t, s^t$

$$\Gamma \equiv \begin{cases} \Delta(s^t) \left\{ \begin{aligned} &u(c_t^A(s^t)) + \beta \sum_{\tau > t} \sum_{s^\tau} \beta^{\tau-t-1} \pi(s^\tau | s^t) u(c^A(s^\tau)) \\ &- [u(y(s^t)) + \beta v_{aut}^A] \geq 0 \end{aligned} \right. \\ Q(\Delta(s^t), s^t) \left\{ \begin{aligned} &u(1 - c_t^A(s^t)) + \beta \sum_{\tau > t} \sum_{s^\tau} \beta^{\tau-t-1} \pi(s^\tau | s^t) u(1 - c^A(s^\tau)) \\ &- [u(1 - y(s^t)) + \beta v_{aut}^B] \geq 0 \end{aligned} \right. \end{cases}$$

$\Delta(s^t)$   
+  $Q(\Delta(s^t), s^t)$   
from  
recursive  
formulation

where autarky values are:

$$v_{aut}^A = \sum_{s \in S} \frac{\pi(s) u(y(s))}{1 - \beta}$$

$$v_{aut}^B = \sum_{s \in S} \frac{\pi(s) u(1 - y(s))}{1 - \beta}$$

Expected utility  
from only eating autarky

↳ For all possible futures, neither household will walk away from contract to autarky

↳ It will always be better to stay in contract than leave

• Now write planner's problem in sequence form:

- Pareto frontier problem in sequence form:

$Q(\Delta)$  } constrained PF

Value promised to (A)

$$= \max_{\{c^A\}_{s^t}} \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t) \beta^t [u(1 - \hat{c}_t^A(s^t)) - u(1 - y(s^t))] \quad \text{(B)}$$

$$\text{s.t.} \quad \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t) \beta^t [\underbrace{u(\hat{c}_t^A(s^t))}_{\text{(A)}} - u(y(s^t))] \geq \Delta$$

$$\{c^A\}_{s^t} \in \Gamma \quad \text{ } \} \text{ sustainable}$$

(A) lifetime utility  $\Delta$  > than autarky

$$c_t^A(s^t) \in [0, 1] \quad \forall t, s^t \quad \text{ } \} \text{ Assume for } u$$

↳ Planner chooses (A) consumption  $\forall t, s^t$  to

maximize (B) lifetime utility s.t. (A) expected

lifetime utility being  $\Delta$  utils greater than value of autarky.

while keeping both Htts in contract.

• Recursive formulation:

$$\textcircled{P} = Q(\Delta, s) = \max_{c, \{\Delta(s')\}_{s'}} \left\{ u(1-c) - u(1-y(s)) + \beta \sum_{s'} \pi(s') Q(\Delta(s'), s') \right\}$$

s.t.

Promise Keeping  $\} \rightarrow u(c) - u(y(s)) + \beta \sum_{s'} \pi(s') \Delta(s') \geq \Delta \quad [\mu]$

Part. (A)  $\} \rightarrow \Delta(s') \geq 0 \quad \forall s' \quad [\beta \pi(s') \lambda(s')]$

Part (B)  $\} \rightarrow Q(\Delta(s'), s') \geq 0 \quad \forall s' \quad [\beta \pi(s') \theta(s')]$

$$c \in [0, 1]$$

•  $Q(\cdot, s)$  is Pareto frontier conditional on realization  $y(s)$

• Call  $\Delta(s)$  continuation value (A) conditional on  $s$

• Thomas - Worrahl (1988) proved:

i)  $\Delta(s) \in [0, \bar{\Delta}(s)] \quad \forall s$

$\hookrightarrow$  Lower: If  $\Delta(s) < 0$  then (A) would exit to autarky

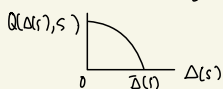
$\hookrightarrow$  Upper:  $\forall s, \bar{\Delta}(s)$  s.t.  $Q(\bar{\Delta}(s), s) = 0$

ii)  $Q(\Delta, s)$  is:

a) Decreasing in  $\Delta$

b) Strictly concave in  $\Delta$

c) continuously differentiable in  $[0, \bar{\Delta}(s)] \quad \forall s$



# Characterizing Optimal Contract

- Concavity of  $Q$  implies  $P$  is convex.

- Lagrangian :

$$\begin{aligned} \mathcal{L} = & u(1-c) - u(1-y(s)) + \beta \sum_{s'} \pi(s') Q(\Delta(s'), s') \\ & + \mu \left( u(c) - u(y(s)) + \beta \sum_{s'} \pi(s') \Delta(s') - \Delta \right) \\ & + \sum_{s'} \beta \pi(s') \lambda(s') (\Delta(s') - 0) \\ & + \sum_{s'} \beta \pi(s') \theta(s') (Q(\Delta(s'), s') - 0) \end{aligned}$$

[Assume Inada]

- FOCs :

$$[c] : -u'(1-c^*) + \mu u'(c^*) = 0$$

$$\begin{aligned} [\Delta(s')] : & \beta \pi(s') Q'(\Delta(s'), s') + \mu \beta \pi(s') \\ & + \beta \pi(s') \lambda(s') + \beta \pi(s') \theta(s') Q'(\Delta(s'), s') = 0 \end{aligned}$$

- ENV :  $Q'(\Delta, s) = -\mu$

- Substituting in FOCs :

$$Q'(\Delta, s) = - \frac{u'(1-\overset{c^*}{c(\Delta, s)})}{u'(c(\Delta, s))}$$

$$Q'(\Delta, s) = (1 + \theta(s)) Q'(\Delta(s'), s') + \lambda(s') \quad \forall s'$$

# Consumption Dynamics

$$Q'(\Delta, s) = - \frac{u'(1 - c(\Delta, s))}{u'(c(\Delta, s))}$$

Decreasing in  $\Delta$ ; Need RHS as well

To be increasing;  $c \uparrow$  in  $\Delta \uparrow$

- Strict concavity of  $Q \Rightarrow c(\Delta, s)$  is increasing in  $\Delta$

↳ Consumption of (A) today is increasing in current promised value to (A)

- Continuity of  $Q' \Rightarrow c(\Delta, s)$  is continuous in  $\Delta$

- $c(\Delta, s)$  increasing in  $\Delta$  and  $\Delta \in [0, \bar{\Delta}(s)]$

$$\Rightarrow c(\Delta, s) \in [\underline{c}(s), \bar{c}(s)] \quad \text{\textcolor{blue}{} } \text{Consumption interval}$$

$$\text{where } \underline{c}(s) \equiv c(0, s)$$

$$\bar{c}(s) \equiv c(\bar{\Delta}(s), s)$$

$$\hookrightarrow \underline{c}(s) \text{ defined by: } Q'(0, s) = - \frac{u'(1 - \underline{c}(s))}{u'(\underline{c}(s))}$$

$\Rightarrow$  If  $c^* = \underline{c}(s)$ , then  $\Delta = 0$  and (A) gets no gain from contract forever

$$\hookrightarrow \bar{c}(s) \text{ defined by: } Q'(\bar{\Delta}(s), s) = - \frac{u'(1 - \bar{c}(s))}{u'(\bar{c}(s))}$$

$\Rightarrow$  If  $c^* = \bar{c}(s)$ ,  $Q(\bar{\Delta}(s), s) = 0$ , and (B) gets no gain

- Define  $c(\Delta, s) = g(Q'(\Delta, s))$ ;

$$\Leftrightarrow Q'(\Delta, s) = g^{-1}(c(\Delta, s))$$

$g$  and  $g^{-1}$  are strictly decreasing

- From FOC:  $g^{-1}(c(\Delta, s)) = (1 + \theta(s')) g^{-1}(c(\Delta(s'), s')) + \lambda(s')$   $\forall s'$   
(A) current consumption (B) tomorrow con. | s'

$\hookrightarrow$  Consumption dynamics in efficient contract governed by participation constraints

$\Rightarrow$  Makes sense! If HH close to leaving to autarky likely compensate more.

- Go through cases for PC (A) and/or (B) binding:

1)  $\lambda(s') = \theta(s') = 0$  (neither PC binds). Then

$$Q'(\Delta, s) = Q'(\Delta(s'), s')$$

$$\Rightarrow g^{-1}(c(\Delta, s)) = g^{-1}(c(\Delta(s'), s'))$$

$$\Rightarrow c(\Delta, s) = c(\Delta(s'), s') \quad (g^{-1} \text{ st. dec})$$

$\Rightarrow c(\Delta(s'), s')$  is independent of  $s'$   
and equals last period's consumption (constant)

$\hookrightarrow$  If promised values  $| s'$  for (A) and (B)

strictly greater than 0, consumption constant across periods

$\Rightarrow$  I.e. Expected lifetime utility  $| s' >$  autarky value  
for both HHs, consumption unchanged

(A binds  $\Rightarrow \Delta(s') = 0$ )

$$2) \lambda(s') > 0 \text{ and } \theta(s') = 0$$

$$Q'(\Delta, s) = Q'(\Delta(s'), s') + \lambda(s')$$

$$g^{-1}(c(\Delta, s)) = g^{-1}(c(\Delta(s'), s')) + \lambda(s')$$

$$\Rightarrow g^{-1}(c(\Delta, s)) > g^{-1}(c(\Delta(s'), s'))$$

$$\Rightarrow c(\Delta, s) < c(\Delta(s'), s')$$

$\hookrightarrow$  Planner increases consumption of (A) just enough so that  $\Delta(s') = 0$ ; (A) indif b/t staying & leaving

$\hookrightarrow$  Since PC(A) binds (i.e.  $\lambda(s') > 0$ )

$$\Rightarrow \Delta(s') = 0$$

$$\Rightarrow c(\Delta(s'), s') = \underbrace{c(s')}_{\approx 0} \quad \} \text{ (A) receives lowest possible consumption to keep in contract}$$

$\Rightarrow$  Consumption tomorrow does not depend on  $\Delta$  or  $s$ ;

$\hookrightarrow$  Contract exhibits "amnesia" when PC(A) binds

$$\text{PC}(\mathbb{B}) \text{ binds} \Rightarrow Q(\Delta(s'), s') = 0$$

$$3) \lambda(s') = 0 \quad \text{and} \quad \theta(s') > 0:$$

$$g^{-1}(c(\Delta, s)) = (1 + \theta(s')) \overbrace{g^{-1}(c(\Delta(s'), s'))}^{< 0}$$

$$\Rightarrow g^{-1}(c(\Delta, s)) < g^{-1}(c(\Delta(s'), s'))$$

$$\Rightarrow c(\Delta, s) > c(\Delta(s'), s') \quad \{ c_A \downarrow, c_R \uparrow$$

↳ Planner increases consumption of  $(\mathbb{B})$  just enough so that  $Q(\Delta(s'), s') = 0$ ;  $(\mathbb{B})$  indif b/t contract/leaving

↳ Again,  $\text{PC}(\mathbb{B})$  binds ( $\theta(s') > 0$ ):

$$Q(\Delta(s'), s') = 0$$

$$\Rightarrow \Delta(s') = \bar{\Delta}(s')$$

$$\Rightarrow c(\Delta(s'), s') = c(\bar{\Delta}(s'), s') = \bar{c}(s')$$

$\Rightarrow$  "Amnesia" again! Consumption tomorrow  $\perp$  of  $\Delta, s$

$$4) \lambda(s') > 0 \quad \text{and} \quad \theta(s') > 0 \quad (\text{both PCs bind})$$

$$\Rightarrow Q(\Delta(s'), s') = 0$$

$$\text{and} \quad \Delta(s') = 0$$

$$\Rightarrow \bar{\Delta}(s') = 0$$

- We can now write down dynamics of optimal contract
- Proposition: Optimal contract consumption dynamics: (given  $c(s)$  and  $s'$ )  
(suppressing  $\Delta$ )

$$c(s') = \begin{cases} \underline{c}(s') & \text{if } c(s) < \underline{c}(s') \quad \Leftrightarrow \quad [\text{PC(A) binds}] \\ \bar{c}(s') & \text{if } c(s) > \bar{c}(s') \quad \Leftrightarrow \quad [\text{PC(B) binds}] \\ c(s) & \text{if } c(s) \in [\underline{c}(s'), \bar{c}(s')] \quad [\text{neither}] \end{cases}$$

Can show

- Note: Can show  $c(s) < \underline{c}(s') \Rightarrow \text{PC(A) binds}$

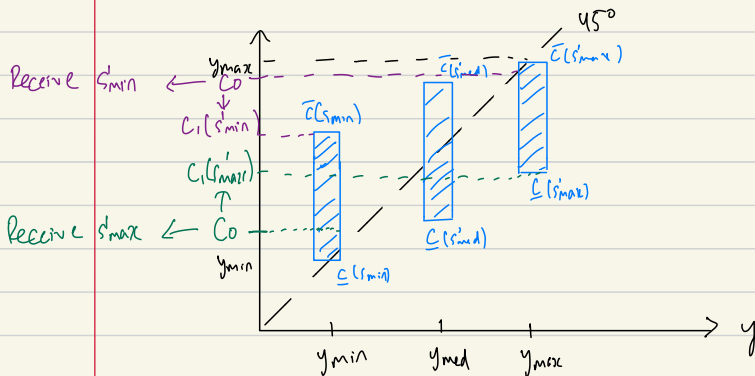
$$\hookrightarrow c(s) < \underline{c}(s') \leq c(s')$$

$$\Rightarrow c(s) < c(s')$$

$$\Rightarrow \text{PC(A) binds}$$

$$\hookrightarrow \text{Symmetric for PC(B)}$$

- Example w/ 3 states:



- Continue characterization of optimal contract

- Proposition : If  $y(s_1) > y(s_2)$

$$\Rightarrow \bar{c}(s_1) > \bar{c}(s_2) \text{ and } \underline{c}(s_1) > \underline{c}(s_2)$$

↳ "Consumption intervals cannot contain each other" but can overlap or be disjoint

- Proposition :  $y(s) \in [\bar{c}(s), \underline{c}(s)]$  ,

$$y(s_{\min}) = \underline{c}(s_{\min}) ,$$

$$y(s_{\max}) = \bar{c}(s_{\max})$$

↳ "Endowments are contained in consumption intervals"

## • First - Best Sustainable Allocation :

↳ Participation constraint of neither (A) nor (B) ever binds

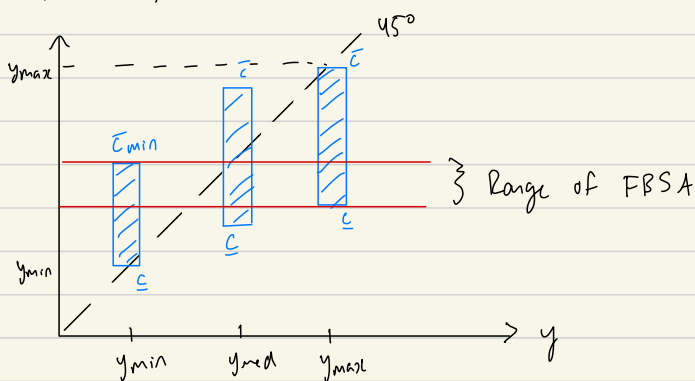
⇒ Consumption always constant; perfect risk sharing

↳ FBSA exists when:

$$\{ [\underline{c}(s), \bar{c}(s)] \}_{j=1}^S = \emptyset$$

$$\Leftrightarrow \bar{c}(s_{\min}) \geq \underline{c}(s_{\max})$$

• Example w/ 3 states:



↳ If FBSA exists, obtains in long run

↳ FBSA exists if

- i)  $\beta$  high enough
- ii) sufficient endowment risk
- iii) Enough curvature of  $u$