

Chari Spring 2022

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Sources: SLP Chap. 13.8
and L-S Chap 28.2 (3rd ed.)

Setup

- Infinite horizon
- Continuum of islands
- Each island receives a shock $z \in \mathbb{Z}$, $|z| < \infty$.
- Shock iid across islands, $z' \sim \pi(z', z)$
- Competitive firms on each island w/ technology
- $f(x, z)$: marginal product on island given
workforce x and shock z
- f : strictly decreasing in x and strictly increasing in z
- $f(0, z) > 0$
- $\exists \bar{x}$ s.t. $f(\bar{x}, z) = 0 \quad \forall z$ } I also assume $f(x, z) \geq 0 \quad \forall x, z$
- Worker preferences:

$$\sum_{t=0}^{\infty} \beta^t C_t$$

Timing:

Worker observes
 (x, z) .
Can stay or
move

Stay: Earn $f(x, z)$ today. Face same
problem tomorrow

Move: Earn 0 today.
Then observe z'
on all islands, and move.
Once at new island face same problem
move or not.

Want to define a stationary competitive equilibrium.

Before we begin the formal analysis, let's think about a general Bellman equation for workers in this problem.

Given current labor force x and shock z on an island, the worker can choose to move to another island or stay on the current.

If the worker chooses to move, we will say that they obtain a fixed value Θ , which represents the expected present value of wages for a worker who chooses to migrate to the best available island. Workers take Θ as given; it is essentially a competitive market price.

If the worker chooses to stay, they earn a wage $f(x, z)$ and the continuation value of staying, $\beta E[v(x', z') | x, z]$, where x' is the labor force on the same island tomorrow.

The Bellman equation for a worker on any island, given (x, z) is:

$$v(x, z) = \max \left\{ \underbrace{\Theta}_{\text{move}}, \underbrace{f(x, z) + \beta E[v(x', z') | x, z]}_{\text{stay}} \right\}$$

We can rewrite this value function by considering the three types of islands Chari asks you to go through. What is useful is that in the rewritten value function, we don't use the conditional expectation.

Three types of islands:

- 1) Positive inflow of immigrants
- 2) Neither inflows nor outflows
- 3) Some or all workers leave

} I go through these in different order than in question.

Note: Each island analysis starts with state (x, z) .

1) Island Type 1: Positive inflow (and all current workers stay)

Chari says: "The wage must be the same in all these islands... Let θ denote this wage."

He also asks "why." This is a bit tricky, as I think θ should be the present value of the wage stream for someone who chooses to move to one of these islands.

This aligns with SLP page 411 and L-S's section 28.2 treatment.

I think the idea of θ being the same in all islands with positive inflow represents the idea that if $\exists$ an island s.t. $\tilde{\theta} > \theta$, then movement to that island would drive $\tilde{\theta}$ to equal θ in equilibrium.

This way, we can think of θ as the value of moving in the model.

Now, what is the Bellman equation for workers on this island?

When would there be a positive inflow into island (x, z) ? When the continuation value on this island conditional on no migration is greater than θ :

$$\theta < \beta \sum_{z'} v(x, z') \pi(z', z)$$

[Remember, movers do not earn today's wage. Thus they only care about the expected present value of future wages on an island. This is reflected in the island's continuation value.]

Workers move to this island until these two values equalize:

$$\theta = \beta \sum_{z'|z} v(x', z') \pi(z', z)$$

$\downarrow$
 x' here

Also note that a positive inflow of agents also implies people currently on the island will stay, assuming $f(x, z) > 0$. Remember, the value of moving is θ and the value of staying is $f(x, z) + \beta \sum_{z'|z} v(x', z') \pi(z', z) = f(x, z) + \theta > \theta$. So value staying $>$ value leaving (θ).

Thus, the Bellman equation for workers on island with positive inflow of immigrants is:

$$v(x, z) = f(x, z) + \theta$$

2) Island Type 2 : No inflows nor outflows

On this island $x = x'$.

Since no one chooses to move here, it must be that:

$$\beta \sum_{z'|z} v(x, z') \pi(z', z) \leq \theta$$

If someone were to move here, then $x' > x$, and

$$\beta \sum_{z'|z} v(x', z') \pi(z', z) \leq \theta \quad \left[v \text{ is non increasing in } x \right]$$

Ignoring edge case of $=$.

Thus no one would choose to move here, as they could move elsewhere and have value of Θ .

Since no one from the island chooses to leave, it must be that the value of staying is greater than or equal to the value of moving (given that $x = x'$).

Using equations, this is:

$$\Theta \leq f(x, z) + \beta \sum_{z'|z} v(x, z') \pi(z', z)$$

Combining these:

$$\beta \sum_{z'|z} v(x, z') \pi(z', z) \leq \Theta \leq f(x, z) + \beta \sum_{z'|z} v(x, z') \pi(z', z)$$

When this holds, $x = x'$.

The Bellman equation for workers on an island with neither inflows nor outflows is:

$$v(x, z) = f(x, z) + \beta \sum_{z'|z} v(x, z') \pi(z', z)$$

Note: $x = x'$

Slightly different than what Chari asked.
Also assuming no arrivals.

3) Island Type 3: Some or All workers leave

If some or all workers leave, it must be that the value of staying if $x=x'$ is strictly less than the value of moving:

$$f(x, z) + \beta \sum_{z'|z} v(x, z') \pi(z', z) < \theta$$

Workers who move will earn the value of moving, θ .

Workers will leave the island (so $x' < x$) until value of staying = value of moving: f is st. dec. in x

$$f(x', z) + \beta \sum_{z'|z} v(x', z') \pi(z', z) = \theta$$

So people leave the island until the value of staying is equal to the value of moving, θ . And movers receive θ by definition.

The Bellman equation for workers on the island where some or all workers leave is:

$$v(x, z) = \theta$$

This completes the analysis of each island.

We now combine the Bellman equations into a common one.

The Bellman equations for workers at each island type, given (x, z) , is below. I include the "condition" that each type of these islands satisfy.

1) Positive inflow:

Condition: $\theta < \beta \sum_{z'|z} v(x, z') \pi(z', z)$

Bellman: $v(x, z) = f(x, z) + \theta$

Intuition: Given the condition above, workers move here until $\beta E[v(x', z') | z] = \theta$.

2) Neither inflow nor outflow:

Condition:

$$\beta \sum_{z'|z} v(x, z') \pi(z', z) \leq \theta \leq f(x, z) + \beta \sum_{z'|z} v(x, z') \pi(z', z)$$

Bellman: $v(x, z) = f(x, z) + \beta \sum_{z'|z} v(\overset{\text{key: } x'=x}{\downarrow} x, z') \pi(z', z)$

Intuition: Here, the labor force is constant $x = x'$.

3) Some or all leave:

Condition: $f(x, z) + \beta \sum_{z'|z} v(x, z') \pi(z', z) < \theta$

Bellman: $v(x, z) = \theta$

Intuition: Given the condition, workers leave (and earn θ) until $f(x', z) + \beta E[v(x', z') | z] = \theta$.

To create the common Bellman equations, begin with the fact that at any island type the Bellman equation (given (x, z)) is just:

$$v(x, z) = \max \left\{ \underbrace{\theta}_{\text{value move}}, \underbrace{f(x, z) + \beta \mathbb{E}[v(x', z') | x, z]}_{\text{value stay}} \right\}$$

Expectation is wrt z' , as well as x' on the same island

We can rewrite $\beta \mathbb{E}[v(x', z') | x, z]$ using our island analysis above.

Conditional on the value of staying being strictly greater than θ , the worker is either on a positive inflow island or a no inflow/outflow island.

- On the positive inflow island :

$$\beta \mathbb{E}[v(x', z') | x, z] = \theta < \beta \sum_{z' | z} v(x, z') \pi(z', z)$$

- On the no inflow/outflow island :

$$\beta \mathbb{E}[v(x', z') | x, z] = \beta \sum_{z' | z} v(x, z') \pi(z', z) \leq \theta$$

Combining these, we see that under this scenario :

$$\beta \mathbb{E}[v(x', z') | x, z] = \min \left\{ \theta, \beta \sum_{z' | z} v(x, z') \pi(z', z) \right\}$$

So our rewritten, common Bellman equation is:

$$v(x, z) =$$

$$\max \left\{ \theta, f(x, z) + \min \left\{ \theta, \beta \sum_{z' \in \mathbb{Z}} v(x, z') \pi(z', z) \right\} \right\}$$

Now, we need to define a stationary equilibrium.

In this definition, we need :

- i) value function and policy function
- ii) An invariant measure
- iii) Market clearing condition
- iv) Price that clears the market (θ)

I write this quickly. It is adapted from Alex's notes.

Let $x \in X$, $z \in \mathbb{Z}$.

$\mathcal{B}(X)$ Borel σ -algebra of X

$P(\mathbb{Z})$ Power set of $\mathbb{Z}$

$$\mathcal{B}(X \times \mathbb{Z}) = \mathcal{B}(X) \times P(\mathbb{Z})$$

A stationary equilibrium consists of :

- i) $v: X \times Z_1 \rightarrow \mathbb{R}$: Value function
- ii) $g: X \times Z_1 \rightarrow X$: policy function describing "end of period work force on the island" (SLP pg. 406) given (x, z) .
I.e. $x' = g(x, z)$
- iii) $\lambda_0: (B(X) \times P(Z_1)) \rightarrow [0, 1]$: Invariant measure "describing distribution of the state over time" (SLP pg. 406)
- iv) θ : market price/value of moving
such that

1) Given θ , v satisfies the Bellman equation and g is the policy function

2) λ_0 is a stationary probability measure:

$$\lambda_0(A, B) = \int Q((x, z), (A, B)) \lambda_0(dx \times dz) \quad \forall (A, B) \in B(X) \times P(Z)$$

and transition function $Q: (X \times Z_1) \times (B(X) \times P(Z_1)) \rightarrow [0, 1]$ satisfies

$$Q((x, z), (A, B)) = \sum_{z' \in B} \begin{cases} \pi(z', z) & \text{if } g(x, z) \in A \\ 0 & \text{o/w} \end{cases}$$

$$\forall (x, z) \in X \times Z_1 \text{ and } \forall (A, B) \in B(X) \times P(Z_1)$$

3) Average island workforce is N (i.e. market clears)

$$D(\theta) = \int x \lambda_0(dx \times dz) = N$$

[Note: N is exogenously given]

For proving existence, ^{of stationary equilibrium} you need to argue v and λ_0 exist and are unique. You also need to show existence of a θ that satisfies the market clearing condition. I point you to SLP chap. 13.8 or the last page of Alex's notes for this.