

Econ 8107 Recitation 4

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1 RCE with a Representative Agent

This section is almost directly from Alex Wurdinger's notes. It is slightly adapted.¹

We have discussed at length the advantages of solving dynamic economic problems recursively instead of in sequence form. In mini 1 and 2 we solved recursive social planner problems. That, however, only works for Pareto efficient economies. Most problems in modern macro are not efficient. In this case we can solve a *Recursive Competitive Equilibrium*.

To illustrate how we can set up/define those equilibria, let us start with the stochastic Neoclassical Growth model. Obviously, here the techniques we are going to use are not necessary, as we can solve this as a social planner problem. It will still be informative however. Remember the sequence problem (SP) of the representative household in the NCGM with Arrow-Debreu securities:

$$\begin{aligned} & \max \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) u(c_t(s^t), l_t(s^t)) \quad \text{s.t.} \\ & \sum_{t=0}^{\infty} \sum_{s^t} p_t(s^t) (c_t(s^t) + k_{t+1}(s^t) - (1 - \delta)k_t(s^{t-1})) \leq \sum_{t=0}^{\infty} \sum_{s^t} (w_t(s^t)l_t(s^t) + r_t(s^t)k_t(s^{t-1})) \end{aligned}$$

Assume s_t follows a Markov process and that the aggregate productivity level $A(s_t)$ is a function of s_t . We know that we can map this competitive equilibria concept to a sequential one with Arrow securities. Thus we want to write the representative household's problem with a sequential budget constraint; since we have a representative agent, we can drop the Arrow Securities (since in equilibrium they will be 0 anyway).

Now the first step to in setting up any recursive problem is to define the appropriate state variables. If we were to rewrite this as a recursive SPP (do this as

¹Alex's notes, 8107 lectures, Ljungqvist and Sargent, and Professor Dirk Krueger's notes are references for these notes. Errors are my own.

practice!), the states would be (K, s) , i.e. the capital stock hold by the representative agent and the realization of today's shock. In a competitive equilibria setting this is not enough.

Households need to forecast next periods prices (wages and interest rate). But these are functions of aggregate capital, which the firms uses as input to production (together with aggregate labor). So we need to add aggregate capital K as a state variable. Since this is an aggregate variable, the agent does not believe it can impact it. We now consider the Bellman in this setting (note that l means labor in this setting):

$$\begin{aligned} v(k, K, s) = \max & \left\{ u(c, l) + \beta \sum_{s'} \pi(s'|s) v(k', K', s') \right\} \\ \text{s.t. } & c + k' - (1 - \delta)k \leq w(K, s)l + r(K, s)k \\ & K' = G(K, s) \\ & l \in [0, 1]; \quad c \geq 0 \end{aligned}$$

First off, notice in the BC that we clearly denote $w(K, s)$ and $r(K, s)$ as functions of aggregate capital and the state. This comes from the FOC of the firm $r(K, s) = A(s)F_k(K, L)$ and $w(K, s) = A(s)F_l(K, L)$. Why are prices then not a function of aggregate labor L ? We will come back to that. Before that, notice that here the distinction between ‘little’ k and ‘big’ K becomes important. Agents take prices as given, they choose k' but have no influence on K' (at least perceived for them).

The evolution of K' is, conversely, governed by the second constraint $K' = G(K, s)$. This is the *aggregate law of motion*. The agents observe this function and know then how today's aggregate capital stock and shock translate into aggregate capital tomorrow. This could also be interpreted as the beliefs or perceptions of the agent about the evolution of the aggregate capital stock (in equilibrium we will require that these beliefs are correct.) So why don't we have something similar in the sequence problem? In the sequence problem the household chooses a sequence of consumption, capital and labor **given** a sequence of prices that clear the market in every period and state. In the recursive formulation, however, the household decides only given today's prices and the law of motion.

We can describe the solutions to the above problem by the policy functions $g_k(k, K, s)$, $g_c(k, K, s)$ and $g_l(k, K, s)$. Notice that the policy functions are functions of the same states as the value function. This is always true. Before we go on, let's just define the sets X and Ω such that $k, K \in X$ and $s \in \Omega$. We can then define the RCE to this economy.

Definition 1 A RCE is defined by a value function $v : X \times X \times \Omega \rightarrow \mathbb{R}$, policy functions g_c, g_k, g_l on the same domain and prices $w(K, s), r(K, s)$ together with a law of motion G such that

1. v attains the maximum of the FE problem given prices and G and the policy functions solve it.
2. Firms maximize profits given prices

$$w(K, s) = A(s)F_l(K, g_l(K, K, s))$$

$$r(K, s) = A(s)F_k(K, g_l(K, K, s))$$

3. Beliefs are consistent with actual evolution of capital

$$G(K, s) = g_k(K, K, s)$$

4. Markets clear

$$g_c(K, K, s) + g_k(K, K, s) = A(s)F(K, g_l(K, K, s)) + (1 - \delta)K$$

Several comments to the above. In conditions 2-4 we see $g_x(K, K, s)$. Here the first element is the aggregate capital stock. This condition means that if we give the agent the aggregate amount of capital, they will produce the aggregate of that variable, e.g. $L = g_l(K, K, s)$. So given the aggregate capital stock, the agent will supply the aggregate labor supply. This formulation depends on us having one type agents.

Using this we can come back to the earlier question on why aggregate labour is not a state variable. The pricing conditions from the firm $r(K, s) = A(s)F_k(K, g_l(K, K, s))$ and $w(K, s) = A(s)F_l(K, g_l(K, K, s))$ clearly only depend on K and s since the aggregate labor decision is a direct function of these two states. Given knowledge about K' , the agent can directly infer L' and thus prices for each future shock.

Finally, condition 3 tells us that we require the aggregate capital stock tomorrow to be consistent with the household capital accumulation decision as the household's asset holdings coincide with the aggregate capital stock. In other words, the belief the household had about how (K, s) translate into K' , given by G is in fact true.

2 RCE with Heterogenous Agents

Last week we defined a complete markets equilibrium with Arrow Securities in an I agent pure exchange setting. Note that all agents are of measure 1.

A pure exchange, sequential markets equilibrium with N agents is:

1.) Household Quantities $\forall i: Z_H^i \equiv \{\{c_t^i(s^t), \{a_{t+1}^i(s_{t+1}, s^t)\}_{s_{t+1}}\}_{s^t}\}_{t=0}^\infty \forall i$

2.) Prices: $P \equiv \{\{\{Q(s_{t+1}|s^t)\}_{s_{t+1}}\}_{s^t}\}_{t=0}^\infty$,

given endowments $\{\{y_t^i(s^t)\}_{s^t} \forall i$, initial distribution of assets $\vec{a}^0 = [a_0, \dots, a_N]$, and borrowing constraints, such that:

i.) Given P , household i chooses Z_H^i to solve:

$$\begin{aligned} \max_{Z_H^i} & \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u^i(c_t^i(s^t)) \pi(s^t) \\ \text{s.t.} \quad & c_t^i(s^t) + \sum_{s_{t+1}|s^t} Q(s_{t+1}|s^t) a_{t+1}^i(s_{t+1}, s^t) \leq y_t^i(s^t) + a_t^i(s^t) \quad \forall t, s^t \\ & a_{t+1}^i(s_{t+1}, s^t) \geq -A_{t+1}^i(s^{t+1}) \quad \forall t, s^t, s_{t+1} \\ & c_t^i(s^t) \geq 0 \quad \forall t, s^t \end{aligned}$$

ii.) Markets clear t, s^t

$$\begin{aligned} \sum_i c_t^i(s^t) & \leq \sum_i y_t^i(s^t) \quad \forall t, s^t \\ \sum_i a_{t+1}^i(s_{t+1}, s^t) & = 0 \quad \forall t, s^t, s_{t+1} \end{aligned}$$

We are now interested in writing this equilibrium **recursively**. This is essentially what you did in Q1 of problem set 3, except with two agents. To do so, we must first give some structure to the stochastic process and endowments. To this end, assume that the stochastic process follows a (first order) Markov chain. Thus, $\pi(s_{t+1}|s^t) = \pi(s_{t+1}|s_t)$. Denote the invariant, or stationary, distribution by π_0 . We also assume that the endowments only depend on today's shock s_t and not the entire history s^t , i.e. $y_t^i(s^t) = y^i(s_t)$.

We want to write agent i 's Bellman equation. What are the state variables in this case? Remember that a state variable generally must satisfy two key requirements:

1. It is predetermined
2. It provides information necessary for the decision-making process

So what are the state variables for the agent in this set up?

- Agent i 's current assets² a
- The current aggregate shock³ s
- The distribution of aggregate assets across agent types $\vec{A} = [A_1, A_2, \dots, A_N]$

²Is predetermined and informs the agent about the RHS of its BC

³Is predetermined and informs the agent of its current endowment $y^i(s)$ and also allows her to know $\pi(s'|s)$.

Why is the **distribution** of aggregate assets a state variable? Because the agent needs to know prices, and prices are a function of the aggregate asset distribution. For example, prices will likely be different in the case that there is high or low asset inequality.

We are almost prepared to write the Bellman equation household i 's problem. First, we need to discuss what constraints the household faces. Obviously, they will face the their budget constraint and a borrowing constraint (which we will assume for simplicity is the same for all agents). The agent also needs another constraint to account for the evolution of the aggregate state $\vec{A}$. Remember, the agents need to know $\vec{A}$ in order to know current prices. Thus, in order to forecast prices the agents must forecast the evolution of the aggregate state.

In the infinite sequence problem the agents chose their optimal sequences *given* the infinite sequence of prices. In the recursive formulation, needs to be provided enough information to forecast the prices.

We will refer to the function $G_i : \mathbb{R}^N \times S \times S \rightarrow \mathbb{R}^N$ as the “beliefs” of agent i about the evolution of the aggregate asset distribution (also called the aggregate law of motion). For simplicity, we will assume $G_i = G$, so that all agents have commonly held beliefs. So a new constraint for the agent is:

$$\vec{A}'(s') = G(\vec{A}, s, s') \quad \forall s' \quad (1)$$

In equilibrium, we will require that these beliefs are correct; this is what we can think of as a “rational expectations” model.

Now, we can write the Bellman equation for household i 's problem:

$$\begin{aligned} v_i(a_i, \vec{A}, s) = \max_{c_i, \{a'_i(s')\}_{s'}} & \left\{ u^i(c_i) + \beta \sum_{s'} v_i[a'_i, \vec{A}'(s'), s'] \pi(s'|s) \right\} \\ \text{s.t.} \quad & c_i + \sum_{s'} Q(s', s, \vec{A}) a'_i(s') \leq y_i(s) + a_i \\ & a'_i(s') \geq -B \quad \forall s' \\ & \vec{A}'(s') = G(\vec{A}, s, s') \quad \forall s' \\ & c_i \geq 0 \\ & s' \sim \pi(s'|s) \end{aligned}$$

Now, define agent i 's optimal policy functions for consumption and assets⁴ as:

$$c = g_c^i(a_i, \vec{A}, s) \quad (2)$$

$$a'(s') = g_{a'(s')}^i(a_i, \vec{A}, s) \quad \forall s' \quad (3)$$

We can now define a recursive competitive equilibrium in this setting.

A recursive competitive equilibrium with Arrow Securities and N agents is:

- 1.) Value functions for each agent: $\{v_i(a_i, \vec{A}, s)\}_{i=1}^N$
- 2.) Optimal policy functions: $\{\{g_{a'(s')}^i(a_i, \vec{A}, s)\}_{s'}, g_c^i(a_i, \vec{A}, s)\}_{i=1}^N$
- 3.) Beliefs about the Aggregate Law of Motion: G
- 4.) Pricing function: $Q(s', s, \vec{A})$
- 5.) Initial distribution of assets: $\vec{a}^0$, such that
 - i.) For all i , given a_0^i , beliefs G , and the pricing function Q , the value function v_i satisfies the Bellman equation and $g_c^i, \{g_{a'(s')}^i\}_{s'}$ are the associated optimal policy functions for household i .
 - ii.) Beliefs G are consistent with actual aggregate law of motion:

$$G(\vec{A}, s, s') = [g_{a'(s')}^1(A_1, \vec{A}, s), g_{a'(s')}^2(A_2, \vec{A}, s), \dots, g_{a'(s')}^N(A_N, \vec{A}, s)] \quad \forall s'$$

- iii.) Markets clear:

$$\sum_i g_{a'(s')}^i(A_i, \vec{A}, s) = 0 \quad \forall s'$$

$$\sum_i g_c^i(A_i, \vec{A}, s) = \sum_i y_i(s)$$

Condition (ii) states that the agent's beliefs about the aggregate law of motion are correct; the distribution of assets actually evolves according to their perceptions about how it would evolve. I.e. the individual policy functions are consistent with the evolution of each household type's *aggregate* assets. By (iii), markets clear in the aggregate. It is important to notice that in condition (ii) and (iii), we use *aggregate* asset A_i of agent type i in the policy functions. We must use aggregate assets A_i for each agent i in these conditions to "impose" equilibrium. Only after household (and/or firm) optimization do we set aggregate variables equal to their agent-level equivalent.

Now that we have seen an RCE in a pure exchange setting, we will study the concept in an economy with production. First, we will define a sequential markets equilibrium with production and N agents (each of measure 1). In this case, uncertainty will come from time endowments.

⁴Note that we have a policy function for Arrow securities for all possible states tomorrow.

A sequential markets equilibrium with production and N agents is:

- 1.) Household Quantities: $Z_H^i \equiv \{\{c_t^i(s^t), \{a_{t+1}^i(s_{t+1}, s^t)\}_{s_{t+1}}, k_{t+1}^i(s^t), n_t^i(s^t)\}_{s^t}\}_{t=0}^\infty \forall i$
- 2.) Firm allocation: $Z^f \equiv \{\{Y_t(s^t), K_t(s^t), N_t(s^t)\}_{s^t}\}_{t=0}^\infty$
- 3.) Prices: $P \equiv \{\{r_t(s^t), w_t(s^t), \{Q(s_{t+1}|s^t)\}_{s_{t+1}}\}_{s^t}\}_{t=0}^\infty$,

given initial distribution of assets $\vec{a}^0 = [a_0, \dots, a_N]$ and capital $\vec{k}^0 = [k_0, \dots, k_N]$, and borrowing constraints, such that:

- i.) Given P , household i chooses Z_H^i to solve:

$$\begin{aligned} & \max_{Z_H^i} \sum_{t=0}^\infty \sum_{s^t} \beta^t u^i(c_t^i(s^t), \bar{n}_t^i(s^t) - n_t^i(s^t)) \pi(s^t) \\ \text{s.t. } & c_t^i(s^t) + k_{t+1}^i(s^t) - (1 - \delta)k_t^i(s^t) + \sum_{s_{t+1}|s^t} Q(s_{t+1}|s^t) a_{t+1}^i(s_{t+1}, s^t) \\ & \leq w_t(s^t) n_t^i(s^t) + r_t(s^t) k_t^i(s^t) + a_t^i(s^t) \quad \forall t, s^t \\ & a_{t+1}^i(s_{t+1}, s^t) \geq -A \quad \forall t, s^t, s_{t+1} \\ & n_t^i(s^t) \in [0, \bar{n}_t^i(s^t)] \\ & c_t^i(s^t), n_t^i(s^t), k_{t+1}^i(s^t) \geq 0 \quad \forall t, s^t \end{aligned}$$

- ii.) Given P , firm chooses Z^f to maximize profits in each period and history:

$$\begin{aligned} & \max Y_t(s^t) - r_t(s^t) K_t(s^t) - w_t(s^t) N_t(s^t) \\ \text{s.t. } & Y_t(s^t) \leq F(K_t(s^t), N_t(s^t)) \quad \forall t, s^t \\ & Y_t(s^t), K_t(s^t), N_t(s^t) \geq 0 \quad \forall t, s^t \end{aligned}$$

- iii.) Markets clear t, s^t

$$\begin{aligned} & \sum_i c_t^i(s^t) + K_{t+1}(s^{t+1}) - (1 - \delta)K_t(s^t) \leq F(K_t(s^t), N_t(s^t)) \quad \forall t, s^t \\ & \sum_i n_t^i(s^t) = N_t(s^t) \quad \forall t, s^t \\ & \sum_i k_t^i(s^{t-1}) = K_t(s^t) \quad \forall t, s^t \\ & \sum_i a_{t+1}^i(s_{t+1}, s^t) = 0 \quad \forall t, s^t, s_{t+1} \end{aligned}$$

Again, as in the previous example, we need to add some more structure to write this problem recursively. Assume that time endowments only depend on the current aggregate shock s_t and not the history of shocks; thus $\bar{n}_t^i(s^t) = \bar{n}^i(s_t)$. Additionally, assume that $\bar{n}_t^i(s_t) = s_t$ for all agents. We will again assume that the evolution of s_t follows a (first order) Markov Chain and that s is strictly positive. So in every period t , all N agents have the same time endowment s_t .

In this case, each agent has the following state variables:

- Agent i 's current Arrow security payout a_i
- Agent i 's current holding of capital k_i
- The aggregate shock/time endowment s
- The distribution of aggregate Arrow security payouts across agent types $\vec{A} = [A_1, A_2, \dots, A_N]$
- The distribution of aggregate capital across agent types $\vec{K} = [K_1, K_2, \dots, K_N]$

The Bellman equation for agent i 's problem is then:

$$\begin{aligned}
 v_i(a_i, k_i, \vec{A}, \vec{K}, s) &= \max_{c_i, \{a'_i(s')\}_{s'}, k'_i, n_i} \left\{ u^i(c_i, s - n_i) + \beta \sum_{s'} v_i[a'_i, k'_i, \vec{A}'(s'), \vec{K}'(s'), s'] \pi(s'|s) \right\} \\
 \text{s.t. } c_i + \sum_{s'} Q(s', s, \vec{A}, \vec{K}) a'_i(s') + k'_i - (1 - \delta)k_i &\leq w(\vec{A}, \vec{K}, s)n_i + r(\vec{A}, \vec{K}, s)k_i + a_i \\
 a'_i(s') &\geq -B \quad \forall s' \\
 \vec{A}'(s') &= G(\vec{A}, \vec{K}, s, s') \quad \forall s' \\
 \vec{K}' &= H(\vec{A}, \vec{K}, s) \\
 n_i &\in [0, s] \\
 c_i, k'_i &\geq 0
 \end{aligned}$$

Note that each agent now has perceptions/beliefs about the aggregate law of motion for the distribution of assets $\vec{A}'(s') = G(\vec{A}, \vec{K}, s, s') \quad \forall s'$ and the distribution of capital $\vec{K}' = H(\vec{A}, \vec{K}, s)$. Note that we don't have an aggregate law of motion for aggregate labor supply, even though we know that wages/rental rates depend on it. As in the representative agent case, this is because aggregate labor supply is itself a function of the distribution of assets and capital. As you will see in the wage and rental rate functions, we account for the optimal labor choices for all agents (and in equilibrium, the sum of labor supplied across agents will be aggregate labor supply).

Define agent i 's optimal policy functions for consumption, Arrow Securities, capital, and labor as:

$$c_i = g_c^i(a_i, k_i, \vec{A}, \vec{K}, s) \quad (4)$$

$$a'_i(s') = g_{a'}^i(a_i, k_i, \vec{A}, \vec{K}, s) \quad \forall s' \quad (5)$$

$$k'_i = g_k^i(a_i, k_i, \vec{A}, \vec{K}, s) \quad (6)$$

$$n_i = g_n^i(a_i, k_i, \vec{A}, \vec{K}, s) \quad (7)$$

A recursive competitive equilibrium with production, Arrow Securities, and N agents is:

- 1.) Value functions for each agent: $\{v_i(a_i, k_i, \vec{A}, \vec{K}, s)\}_{i=1}^N$
- 2.) Optimal policy functions (on the same domain as the value functions): $\{\{g_{a'(s')}^i\}_{s'}, g_c^i, g_k^i, g_n^i\}_{i=1}^N$
- 3.) Beliefs about the Aggregate Law of Motion for distribution of Arrow securities G and distribution of capital H
- 4.) Pricing functions: $Q(s', s, \vec{A}, \vec{K}), w(\vec{A}, \vec{K}, s), r(\vec{A}, \vec{K}, s)$
- 5.) Initial distributions of Arrow securities and capital: $\vec{a}^0, \vec{k}^0$ such that
 - i.) For all i , given a_0^i, k_0^i , beliefs G, H and the pricing functions Q, w, r , the value function v_i satisfies the Bellman equation and $g_c^i, g_a^i, g_k^i, g_n^i$ are the associated optimal policy functions.
 - ii.) Factor prices are such that firm maximizes profits (note again that each agent is of measure 1):

$$w(\vec{A}, \vec{K}, s) = F_N(\sum_i K_i, \sum_i g_n^i(A_i, K_i, \vec{A}, \vec{K}, s))$$

$$r(\vec{A}, \vec{K}, s) = F_K(\sum_i K_i, \sum_i g_n^i(A_i, K_i, \vec{A}, \vec{K}, s))$$

- iii.) Belief functions G and H are consistent with actual aggregate law of motion (i.e. generated by policy functions):

$$G(\vec{A}, \vec{K}, s, s') = [g_{a'(s')}^1(A_1, K_1, \vec{A}, \vec{K}, s), g_{a'(s')}^2(A_2, K_2, \vec{A}, \vec{K}, s), \dots, g_{a'(s')}^N(A_N, K_N, \vec{A}, \vec{K}, s)] \quad \forall s'$$

$$H(\vec{A}, \vec{K}, s) = [g_k^1(A_1, K_1, \vec{A}, \vec{K}, s), g_k^2(A_2, K_2, \vec{A}, \vec{K}, s), \dots, g_k^N(A_N, K_N, \vec{A}, \vec{K}, s)]$$

- iv.) Markets clear:

$$\sum_i g_{a'(s')}^i(A_i, K_i, \vec{A}, \vec{K}, s) = 0 \quad \forall s'$$

$$\sum_i g_c^i(A_i, K_i, \vec{A}, \vec{K}, s) + \sum_i g_k^i(A_i, K_i, \vec{A}, \vec{K}, s) - (1-\delta)(\sum_i K_i) = F(\sum_i K_i, \sum_i g_n^i(A_i, K_i, \vec{A}, \vec{K}, s))$$

Again, note that in conditions (ii)-(iv), we replace “little” (household-level) a_i and k_i with “big” (aggregate) A_i and K_i in the policy functions. This is very important, as it imposes equilibrium.

3 Incomplete Markets

Now that we have viewed RCEs in a complete markets setting, let's go to an incomplete market setting. Unlike the previous section where we assumed there were multiple types of agents, here we will assume that there is a continuum of households with total mass of 1. In general, these households may be identical before any risk is realized (households are *ex-ante identical*), but will certainly become heterogeneous (due to differences in the realizations of their income across time) after uncertainty is realized (households are *ex-post heterogeneous*).

What makes our example below an “incomplete” market? Instead of being able to trade Arrow securities, the household can only trade a single security/asset. As the writer of the model, we are essentially “closing” one-step-ahead contingent claims markets. Because agents can no longer trade assets for *every* date and possible state, markets are no longer complete - they are incomplete.

3.1 Generalized Huggett Model

This summarizes Dirk Krueger’s Chapter 10.3 notes, which is a generalized version of Huggett (1993). This can be extended to an economy with production, which is what Aiyagari (1994) does. Alex Wurdinger’s notes cover Aiyagari.

There is a continuum of agents of total mass 1. Agents have identical stochastic endowment *processes* that follow an m-state Markov chain (but remember that agents will have different income *realizations*). Possible values of the endowment are $Y = \{y_1, \dots, y_M\}$ and $\pi(y'|y)$ is the probability of endowment y' tomorrow given today’s endowment is y . Each household faces idiosyncratic risk because of the stochastic income process.

The law of large numbers holds by assumption. Thus $\pi(y'|y)$ also represents the percent of households who transition from y to y' . Denote Π as the unique stationary distribution for the Markov chain π . We also assume that the initial endowment y_0 is drawn from Π . Since the initial income process is drawn from the stationary distribution, we know that at any point in time the fraction of households with endowment y is $\Pi(y)$.

A quick aside on stationary/invariant distribution of a Markov chain. We call a Π the stationary distribution of a Markov chain with a transition matrix $\mathcal{P}$ if the following is satisfied:

$$\Pi = \Pi \mathcal{P}$$

Because of this stationary distribution assumption, the total endowment at any point of time is constant:

$$Y = \sum_y y \Pi(y)$$

In other words, there is no aggregate uncertainty even though there is significant idiosyncratic uncertainty. Each household faces the following problem (given (a_0, y_0)):

$$\begin{aligned} & \max_{\{c_t(y^t), a_{t+1}(y^t)\}_{y^t} \}_{t=0}^{\infty}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t c_t(y^t) \right] \\ & \text{s.t. } c_t(y^t) + a_{t+1}(y^t) = (1 + r_t) a_t(y^{t-1}) + y_t \quad \forall t, y^t \end{aligned}$$

$$a_{t+1}(y^t) \geq -B \quad \forall t, y^t$$

Now, to define the problem recursively we need to define the households' individual state variables and the aggregate state variables.

- Individual state variables: Assets a and endowment y
- Aggregate state variable: Cross-sectional distribution over household characteristics $\Phi(a, y)$

Now we will define some objects so we can define the measurable space for measures Φ .

- Set of possible asset holdings: $A = [-B, \infty]$
- Set of possible endowments: Y
- Power set of Y : $\mathcal{P}(Y)$
- Borel σ -algebra of A : $\mathcal{B}(A)$
- All possible asset/endowment pairs: $Z = A \times Y$
- $\mathcal{B}(Z) = \mathcal{P}(Y) \times \mathcal{B}(A)$
- Set of all probability measures on measurable space $(Z, \mathcal{B}(Z))$: $\mathcal{M}$

Note that we will require that $\Phi \in \mathcal{M}$. The recursive formulation of the household's problem is then:

$$v(a, y, \Phi) = \max_{a, a'} u(c) + \sum_{y'} \pi(y'|y) v(a', y', \Phi')$$

$$\text{s.t. } c + a' = y + (1 + r(\Phi))a$$

$$\phi' = H(\Phi)$$

$$a' \geq -B; \quad c \geq 0$$

The interest rate is now a function of the probability measure/distribution Φ . We also see the aggregate law of motion, or what we referred to as beliefs, $H : \mathcal{M} \rightarrow \mathcal{M}$. Given the distribution Φ today, the households must be able to forecast Φ' in order to forecast the interest rate tomorrow.

I will now use Professor Krueger's definition of a RCE in this setting. This is written nearly exactly as in his notes.

A recursive competitive equilibrium is a value function $v : Z \times \mathcal{M} \rightarrow \mathbb{R}$, policy functions $a' : Z \times \mathcal{M} \rightarrow \mathbb{R}$ and $c : Z \times \mathcal{M} \rightarrow \mathbb{R}$, a pricing function $r : \mathcal{M} \rightarrow \mathbb{R}$ and

an aggregate law of motion $H : \mathcal{M} \rightarrow \mathcal{M}$ such that

- 1.) Given $r()$, v satisfies the household's Bellman equation and a', c are the associated optimal policy functions.
- 2.) Markets clear $\forall \Phi \in \mathcal{M}$:

$$\int c(a, y; \Phi) d\Phi = \int y d\Phi$$

$$\int a'(a, y; \Phi) d\Phi = 0$$

- 3.) The aggregate law of motion H is consistent with (generated by) the Markov process π and the policy function a' ; i.e. for all $(\mathcal{A}, \mathcal{Y}) \in \mathcal{B}(Z)$

$$\Phi'(\mathcal{A}, \mathcal{Y}) = (H(\Phi))(\mathcal{A}, \mathcal{Y}) = \int Q((a, y), (\mathcal{A}, \mathcal{Y})) d\Phi$$

where $Q : Z \times \mathcal{B}(Z) \rightarrow [0, 1]$ is a transition function defined by:

$$Q((a, y), (\mathcal{A}, \mathcal{Y})) = \sum_{y' \in \mathcal{Y}} \begin{cases} \pi(y'|y) & \text{if } a'(a, y; \Phi) \in \mathcal{A} \\ 0 & \text{if } a'(a, y; \Phi) \notin \mathcal{A} \end{cases}$$

for all $(a, y) \in Z$ and all $(\mathcal{A}, \mathcal{Y}) \in \mathcal{B}(Z)$ This completes the definition of an RCE for this setting.

First, note that the market clearing condition (ii) must hold for all measures on the measurable space.

Condition (3) is our consistency requirements from the previous examples. However, instead of having beliefs/perceptions about the evolution of a single aggregate K or a vector of K_i 's, the agents need to be able to forecast the evolution of a probability measure over asset-endowment pairs. Since each household is completely characterized by their asset holding a and endowment y , this is identical to tracking the how the distribution of households period-by-period.

We use a transition function Q because we need to know how households (who are characterized by their asset and endowment) move over time. The transition function tells us the probability a household with assets a and endowment y today obtains assets a' in $\mathcal{A}$ and endowment y' in $\mathcal{Y}$ tomorrow. We can write Q in a more familiar format using our usual conditional probability notation:

$$Q((a, y), (\mathcal{A}, \mathcal{Y})) \equiv \mathbb{P}(a' \in \mathcal{A}, y' \in \mathcal{Y} | a, y)$$

Professor Krueger provides an intuitive example where $\mathcal{Y} = \{y_1\}$. The probability a household with (a, y) today ends up with assets $a' \in \mathcal{A}$ and $y' = y_1$ tomorrow is

equal to $\pi(y_1|y)$ if the household's optimal choice of a' is in $\mathcal{A}$, i.e. $a'(a, y; \Phi) \in \mathcal{A}$. Otherwise, if $a'(a, y; \Phi) \notin \mathcal{A}$, the probability the household ends up with y_1 and $a' \in \mathcal{A}$ is 0. This household will never optimally choose assets in $\mathcal{A}$, so they will never end up in the set $\{y_1\} \times \mathcal{A}$.

To formulate $\Phi'(\mathcal{A}, \mathcal{Y})$ we need to integrate Q with respect to the current measure Φ .

$$\Phi'(\mathcal{A}, \mathcal{Y}) = (H(\Phi))(\mathcal{A}, \mathcal{Y}) = \int Q((a, y), (\mathcal{A}, \mathcal{Y})) d\Phi$$

The integral essentially sums over all (a, s) pairs (i.e. households) and “weights” $Q((a, y), (\mathcal{A}, \mathcal{Y}))$ by the mass of households at each (a, s) pair. This is explicitly done in the LS simplified model of Huggett (see the end of these notes), where endowments and assets both lie on a discrete grid.

Now we will define a stationary RCE. The key part of a stationary RCE is that the measure does not change over time; $\Phi_{t+1} = \Phi_t = \Phi^* \quad \forall t$. Since interest rates and the policy functions were functions of the measure Φ in the general RCE, we see that prices and decision rules are constant over time in a *stationary* RCE. This makes the equilibrium much easier to compute. Again, this definition is basically directly from Professor Krueger's notes:

A **stationary recursive competitive equilibrium** is a value function $v : Z \rightarrow \mathbb{R}$, policy functions $a' : Z \rightarrow \mathbb{R}$ and $c : Z \rightarrow \mathbb{R}$, an interest rate r^* and a probability measure Φ^* such that

- 1.) Given r^* , v satisfies the household's Bellman equation and a', c are the associated optimal policy functions.
- 2.) Markets clear:

$$\begin{aligned} \int c(a, y) d\Phi^* &= \int y d\Phi^* \\ \int a'(a, y) d\Phi^* &= 0 \end{aligned}$$

- 3.) For all $(\mathcal{A}, \mathcal{Y}) \in \mathcal{B}(Z)$

$$\Phi^*(\mathcal{A}, \mathcal{Y}) = \int Q((a, y), (\mathcal{A}, \mathcal{Y})) d\Phi^*$$

where $Q : Z \times \mathcal{B}(Z) \rightarrow [0, 1]$ is a transition function defined as before. This completes the definition.

Note that prices, the value function, and the policy functions are no longer functions of all possible probability measures $\Phi \in \mathcal{M}$. This is because there is only one measure that matters, the stationary measure Φ^* . Since this measure does not change over time, we no longer have to index prices, the value function, or policy functions

by it. Note that the consistency requirement now says that the equilibrium measure must “reproduce itself.” See that Φ^* is on both the RHS and LHS. So if we integrate over the transition function Q with respect to Φ^* , we end up with Φ^* again.

3.2 Simplified Huggett Model

For this section, I will follow Chapter 18 of LS. This is a simplified version of what was covered in the previous section. There is a unit measure of households. Labor income follows an m -state Markov Chain; thus $S = \{s_1, \dots, s_m\}$. Denote the transition matrix for this Markov Chain by $\mathcal{P}$. Labor income at period t is ws_t if s_t is the realization of the process. For example, we could assume that $m = 2$ and $S = \{0, 1\}$; then we would view $s_t = 0$ as if the household was unemployed and $s_t = 1$ if the household is employed. For simplicity, we will also assume that assets are constrained to a grid $\mathcal{A} = [a_1 < a_2 < \dots < a_n]$. The household’s problem in sequence form, given prices (w, r) and initial values (s_0, a_0) is:

$$\begin{aligned} & \max_{\{c_t(s^t), a_{t+1}(s^t)\}_{s^t}^\infty} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t c_t(s^t) \right] \\ \text{s.t. } & c_t(s^t) + a_{t+1}(s^t) = (1 + r)a_t(s^{t-1}) + ws^t \quad \forall t, s^t \\ & a_{t+1}(s^t) \in \mathcal{A} \quad \forall t, s^t \end{aligned}$$

Notice that we no longer have Arrow securities in this setup. If a household chooses a value of assets $a_{t+1}(s^t)$ in period t , history s^t , they will be paid $(1 + r)a_{t+1}(s^t)$ in period $t + 1$, history s^{t+1} regardless of the realization of s_{t+1} .

For this partial equilibrium model, the states are the households current asset holding a and their income shock $s_i \in S$. The Bellman equation for this problem is:

$$v(a, s_i) = \max_{a' \in \mathcal{A}} \{u[(1 + r)a + ws_i - a'] + \beta \sum_{j=1}^m \mathcal{P}(i, j)v(a', s_j)\}$$

Denote the optimal policy function by $a' = g(a, s)$. We are now interested in studying the distribution of wealth-employment pairs (a_t, s_t) across time. Let $\lambda_t(a, s) = \mathbb{P}(a_t = a, s_t = s)$, the unconditional distribution of (a_t, s_t) . The optimal policy function along with the transition matrix induce a law of motion for this distribution:

$$\lambda_{t+1}(a', s') = \sum_a \sum_s \lambda_t(a, s) \mathbb{P}(s_{t+1} = s' | s_t = s) \cdot \mathcal{I}(a', s, a) \quad (8)$$

where $\mathcal{I}(a', s, a)$ is an indicator function for whether a' is optimal given (a, s) .

$$\mathcal{I}(a', s, a) = \begin{cases} 1 & \text{if } a' = g(a, s) \\ 0 & \text{otherwise} \end{cases}$$

Note that:

$$\mathbb{P}(s_{t+1} = s' | s_t = s) \cdot \mathcal{I}(a', s, a) = \mathbb{P}((s_{t+1} = s', a'_{t+1} = a') | (s_t = s, a_t = a))$$

So we can write (8) like we wrote the evolution of the probability measure in the previous section. We have a transition function $\mathbb{P}((s_{t+1} = s', a'_{t+1} = a') | (s_t = s, a_t = a))$ (which corresponds to Q in the previous section), and we take the weighted sum (using λ_t) over all (a, s) pairs to obtain tomorrow's distribution λ_{t+1} .

$$\lambda_{t+1}(a', s') = \sum_a \sum_s \mathbb{P}((s_{t+1} = s', a'_{t+1} = a') | (s_t = s, a_t = a)) \lambda_t(a, s) \quad (9)$$

Another way to write the two equations above is:

$$\lambda_{t+1}(a', s') = \sum_s \sum_{\{a: a' = g(a, s)\}} \lambda_t(a, s) \mathcal{P}(s, s') \quad (10)$$

We call a probability distribution λ for which $\lambda = \lambda_{t+1} = \lambda_t$ a *stationary distribution*.

$$\lambda(a', s') = \sum_s \sum_{\{a: a' = g(a, s)\}} \lambda(a, s) \mathcal{P}(s, s') \quad (11)$$

In words, the wealth-employment distribution does not change over time; it is a time invariant distribution. We can view the stationary distribution in two different ways:

1. Percent of time an infinitely lived agent is in state (a, s)
2. The percent of households in state (a, s) at any point in time

Using this stationary distribution and the policy function that induces it, we can calculate the average assets in an economy (for a fixed r):

$$E(a)(r) = \sum_{a, s} g(a, s) \lambda(a, s)$$

Now we are ready prepared to introduce the Huggett (1993) model.

- There is a continuum of households with access to risk free loans
- Endowments are governed by a Markov chain $(\mathcal{P}, S)$
- Households can borrow or lend at constant interest rate r
- Asset grid $\mathcal{A} = [a_1 < a_2 < \dots < a_n]$, where $a_1 = -\phi$

The solution to the household's problem is an optimal asset policy function $a' = g(a, s)$ that induces a stationary distribution λ (along with the corresponding consumption policy function). We will now define a **stationary** recursive equilibrium in this setting. Note that prices are constant in this setting, since the distribution does not change over time; this is what makes a stationary equilibrium special.

A stationary recursive equilibrium in this setting is a value function v , optimal policy functions $g(a, s), c(a, s)$, a constant interest rate r , and stationary distribution λ that satisfy:

- i.) Given r , the value function v satisfies the Bellman equation and $g(a, s), c(a, s)$ are the associated optimal policy functions.
- ii.) $\lambda(a, s)$ is the invariant distribution of the Markov chain on (a, s) induced by the Markov Chain $(\mathcal{P}, S)$ and the optimal policy function $g(a, s)$:

$$\lambda(a', s') = \sum_s \sum_{\{a: a'=g(a, s)\}} \lambda(a, s) \mathcal{P}(s, s')$$

- iii.) Markets clear:

$$\begin{aligned} \sum_{a, s} g(a, s) \lambda(a, s) &= 0 \\ \sum_{a, s} c(a, s) \lambda(a, s) &= \sum_{a, s} s \lambda(a, s) \end{aligned}$$