

$$1) \quad a) \quad v(b, y) = \max_{c, b'} \left\{ u(c) + \beta \underbrace{\mathbb{E}[v(b', y') | y]}_{\text{blue}} \right\}$$

$$\text{s.t.} \quad c + qb' \leq b + y$$

$$b' \geq \psi$$

$$c \geq 0$$

(b) FOCs: λ on BC ; θ on BC

$$[c]: \quad u'(c) = \lambda$$

$$[b']: \quad \beta \sum_{y'} \pi(y' | y) v_{b'}(b', y') - \lambda q + \theta = 0$$

$$\Rightarrow \quad \lambda q = \beta \sum_{y'} \pi(y' | y) v_{b'}(b', y') + \theta$$

Combining FOCs:

$$u'(c) \geq \frac{\beta}{q} \mathbb{E}[v_{b'}(b', y') | y] ; = \text{ if } b' > \psi$$

Envelope:

$$v_b(b, y) = u'(c)$$

$$\Rightarrow v_{b'}(b', y) = u'(c')$$

$$\text{Euler: } u'(c) \geq \frac{\beta}{q} E[u'(c') | y] \\ [= \text{ if } b' \geq \Psi]$$

Need Euler to characterize solution. $c + qb' \leq b + y$; $b' \geq \Psi$; $c \geq 0$; and

c) WTS: g_b non-decreasing for each y .

Fix $y \in Y$.

Suppose $b_1 > b_2$ but $g_b(b_1, y) < g_b(b_2, y)$

First, note that since $b' \geq \Psi$, then

$$g_b(b_2, y) > g_b(b_1, y) \geq \Psi$$

$$\Rightarrow g_b(b_2, y) > \Psi.$$

Thus, we know that $\Theta = 0$ given (b_2, y) from complementary slackness (i.e. $\Theta \cdot (b' - \Psi) = 0$).

Since u is strictly concave and setup

is "nice" enough, we know that

v "inherits" strict concavity from u .
(Thanks SLP!)

Since v is strictly concave in b' , then $v_{b'}$ will be strictly decreasing in b' (for each y'):
 $\Rightarrow v_{b'}(g_b(b_1, y), y') > v_{b'}(g_b(b_2, y), y') \quad \forall y' \in Y$
 $\Rightarrow \pi(y'|y) v_{b'}(g_b(b_1, y), y') > \pi(y'|y) v_{b'}(g_b(b_2, y), y') \quad \forall y' \in Y$

Summing across all $y' \in Y$ and multiplying by $\frac{B}{q} > 0$:

$$\textcircled{*} \quad \frac{B}{q} \mathbb{E}[v_{b'}(g_b(b_1, y), y') | y] > \frac{B}{q} \mathbb{E}[v_{b'}(g_b(b_2, y), y') | y]$$

From FOCs, we know:

$$u'(b_1 + y - q g_b(b_1, y)) = \frac{B}{q} \mathbb{E}[v_{b'}(g_b(b_1, y), y') | y]$$

$$u'(b_2 + y - q g_b(b_2, y)) = \frac{B}{q} \mathbb{E}[v_{b'}(g_b(b_2, y), y') | y]$$

since $g_b(b_2, y) > \psi$

Combining this with $\textcircled{*}$, we have:

$$u'(b_1 + y - q g_b(b_1, y)) > u'(b_2 + y - q g_b(b_2, y))$$

u' is strictly decreasing:

$$b_1 + y - q g_b(b_1, y) < b_2 + y - q g_b(b_2, y)$$

$$\underbrace{0}_{\text{original assump.}} < b_1 - b_2 < \underbrace{q(g_b(b_1, y) - g_b(b_2, y))}_{> 0} < \underbrace{0}_{\text{original assump.}}$$

Thus we have $0 > b_1 - b_2 > 0$, a contradiction.

So g_b is nondecreasing in b .

d) Define an RCE:

↳ Individual states: b_i, y_i

↳ Aggregate state: $\Phi(b, y)$ distribution over bonds and income

First, define objects:

Let $\cdot b \in A = [\psi, \infty)$

with Borel algebra: $\mathcal{B}(A)$

$\cdot y \in Y$

with Borel alg. (Power set): $\mathcal{P}(Y)$

$\cdot A \times Y$

with $\mathcal{B}(A \times Y) = \mathcal{B}(A) \times \mathcal{P}(Y)$

$\cdot M = (A \times Y, \mathcal{B}(A \times Y))$ the measurable space

$\cdot \mathcal{M}$ set of all probability measures on M .

$\cdot \Phi \in \mathcal{M}$ is probability measure

$$v(b, y, \Phi) = \max_{b', c} \left\{ u(c) + \beta \sum_{y'} \pi(y'|y) v(b', y', \Phi') \right\}$$

$$\text{s.t. } c + q(\Phi) b' \leq b + y$$

$$\Phi' = H(\Phi)$$

$$b' \geq \psi; c \geq 0$$

Policy functions:

$$g_c(b, y, \Phi); g_b(b, y, \Phi)$$

An RCE is:

- i) Value f^n : $v: A \times Y \times \mathcal{M} \rightarrow \mathbb{R}$
- ii) Policy f^n s: g_b, g_c on same domain
- iii) Pricing f^n : $q: \mathcal{M} \rightarrow \mathbb{R}$
- iv) Agg. law of motion: $H: \mathcal{M} \rightarrow \mathcal{M}$ s.t.

1) Given q & H , v satisfies H 's Bellman equation and g_b, g_c are associated policy f^n s

2) Markets clear $\forall \Phi \in \mathcal{M}$

$$\int g_c(b, y, \Phi) d\Phi = \int y d\Phi = \int y d\Pi$$

$$\int g_b(b, y, \Phi) d\Phi = 0$$

3) Agg. law of motion H is consistent w/ Markov Process
and policy f^n $g_b: \forall (A, Y) \in \mathcal{B}(A \times Y)$ and $(b, y) \in A \times Y$

$$\Phi'(A, Y) = (H(\Phi))(A, Y) = \int Q((b, y), (A, Y)) d\Phi$$

$$\text{where } Q((b, y), (A, Y)) = \sum_{y' \in Y} \begin{cases} \pi(y') y & \text{if } g_b(b, y, \Phi) \in A \\ 0 & \text{if } g_b(b, y, \Phi) \notin A \end{cases}$$

e) Stationary: $\cdot \Phi^*$ is stationary $\Rightarrow$ doesn't change over time,
so $\Phi_t = \Phi^* \forall t$

- $\cdot$ Price doesn't change over time as $q(\Phi_t) = q(\Phi^*) \forall t$
- $\cdot$ Value f^n , policy f^n , pricing f^n no longer need have $\mathcal{M}$ as part of their domain
- $\cdot$ Don't need H as constraint for agent
- $\cdot$ Φ "reproduces" itself given Markov process + policy f^n s

f) Can $a_t = b_t + y_t$ be only state variable in Bellman equation ($q_t = q$ i.e. stationary).

No. You could if you assume y_t is iid.

Need to know y_t to form expectation over y_{t+1} , since y follows a Markov process.

• Bellman equation assuming y_t is iid and

using $a_t + y_t$ as state variable:

$$v(a; q) = \max_{c, b'} \left\{ u(c) + \beta \underbrace{\sum_{y'} \pi(y') v(\underbrace{b' + y'}_{a'}; q)}_{\text{Not conditioning on } y} \right\}$$

$$\text{s.t. } c + qb' \leq a$$

$$b' \geq \psi$$

$$c \geq 0$$

Note that you could still use b and y as state variables if y was iid.

$a =$ However, since the agent needs to know only $(b+y)$ and not b and y separately, it is unnecessary.

State variables tell agents what they need to know to make decisions today. When y is iid, agent only needs to know $b+y$.

2) $w_t = \phi^t w$ after t periods on the job.

$$E\left[\sum_{t=0}^{\infty} \beta^t y_t\right]$$

a) Unemployed worker with current wage offer w :

$$V(w) = \max_{A, R} \{V^A(w), V^R\}$$

$$\text{Accept: } V^A(w) = w + \phi\beta w + \phi^2\beta^2 w + \dots$$

$$= \frac{w}{1-\phi\beta} \quad \left. \vphantom{\frac{w}{1-\phi\beta}} \right\} \text{Assumed } \phi\beta < 1$$

$$\text{Reject: } V^R = c + \beta \int_0^B V(w') dF(w')$$

$$\Rightarrow V(w) = \max_{A, R} \left\{ \frac{w}{1-\phi\beta}, c + \underbrace{\beta \int_0^B V(w') dF(w')}_{E[V(w')]} \right\}$$

b) Resv. wage:

i) Define operator $T(F)(w) = \max_{A, R} \left\{ \frac{w}{1-\phi\beta}, c + \beta E[F(w')] \right\}$

ii) Maps from $C(X) \rightarrow C(X)$ (cont. bounded)

iii) Satisfies Blackwell's: Monotonicity and Discounting

$\Rightarrow T$ is a contraction; $\exists!$ cont, bounded v^∞ that satisfies FE

$$V^A(w) = \frac{w}{1-\phi\beta} \quad \text{st. increasing and continuous in } w$$

V^R is constant in w .

$$\text{Argue } \underbrace{V^A(0)}_{\text{Reject}} < V^R < \underbrace{V^A(B)}_{\text{Accept}}$$

By IVT: $\exists w^*$ s.t. $V^A(w^*) = V^B$

$$\Rightarrow V(w) = \begin{cases} \frac{w}{1-\beta} & \text{if } w > w^* \\ \frac{w^*}{1-\beta} = V^B & \text{if } w \leq w^* \end{cases}$$

$$\begin{aligned} c) \frac{w^*}{1-\beta} &= c + \beta E[V(w')] = \beta \int_0^B V(w') dF(w') \\ &= \beta \left[\int_0^{w^*} \frac{w^*}{1-\beta} dF(w) + \int_{w^*}^B \frac{w}{1-\beta} dF(w) \right] \end{aligned}$$

Multiply both sides by $\frac{1-\beta}{1-\beta}$

$$\Rightarrow \frac{w^*}{1-\beta} = \underbrace{\frac{c(1-\beta)}{1-\beta}}_{\hat{c}} + \beta \left[\int_0^{w^*} \frac{w^*}{1-\beta} dF(w) + \int_{w^*}^B \frac{w}{1-\beta} dF(w) \right]$$

$\Rightarrow$ McCall model with benefit $\hat{c}$.

Know that w^* is increasing in benefit (see Tim's notes).

$\hat{c}$ is decreasing in ϕ

$\Rightarrow w^*$ is decreasing in ϕ

$$\text{Algebra: } w^* - \frac{c(1-\beta)}{1-\beta} = \frac{\beta}{1-\beta} \int_{w^*}^B w' - w^* dF(w')$$

Sketch of answer: {

One option is comparative statics argument: In eqn above must hold.

$\phi \uparrow$ LHS $\uparrow$; LHS $>$ RHS if w^* does not change

$\Rightarrow w^* \downarrow \Rightarrow$ LHS $\downarrow$ and RHS $\uparrow$. Thus w^* needs to $\downarrow$ to be in eqn.

$\Rightarrow \phi \uparrow \Rightarrow w^* \downarrow$

$\downarrow$ since $\int_{w^*}^B$ expands and $w - w^*$ increases

Could also use Leibniz to be more thorough.

3) a) → Note: This is the two agent case in the RCE Recitation Notes. 2 agents w/ different prefs. Both of measure 1.

Output: $A(s) F(K_t, L_t)$

Let $\tilde{K}$ and $\tilde{A}$ be sets possible values Capital and Arrow securities

$$\text{Let } \vec{K} = [K_1, K_2]$$

$$\vec{A} = [A_1, A_2]$$

$$v_i(a_i, k_i, \underbrace{\vec{K}}_{\text{Aggregate state of economy}}, \underbrace{\vec{A}}_{\text{Individual state}}, s) = \max_{c, l, a'(s'), k'} \left\{ u^i(c, l) + \beta E[v^i(a'(s'), k_i, \vec{K}, \vec{A}(s'), s')] \right\}$$

$$\text{s.t. } c_i + \sum_{s'} q(s, s', \vec{K}, \vec{A}) a'_i(s') + k'_i$$

$$\leq w(s, \vec{A}, \vec{K}) l_i + r(s, \vec{A}, \vec{K}) k_i$$

$$\begin{aligned} \vec{K}' &= H(\vec{A}, \vec{K}, s) \\ \vec{A}'(s') &= G(\vec{A}, \vec{K}, s, s') \quad \forall s' \\ a'(s') &= -B \quad \forall s' \end{aligned}$$

$$l_i \in [0, 1], \text{ non-neg}$$

$$\begin{aligned} \text{Policy fns: } & g_c^i(a_i, k_i, \vec{A}, \vec{K}, s) \\ & g_{a'(s')}^i(a_i, k_i, \vec{A}, \vec{K}, s) \quad \forall s' \\ & g_k^i(a_i, k_i, \vec{A}, \vec{K}, s) \\ & g_l^i(a_i, k_i, \vec{A}, \vec{K}, s) \end{aligned}$$

RCE.

$$\text{i) Value f}^n: v_i: \tilde{A} \times \tilde{K} \times \tilde{A}^2 \times \tilde{K}^2 \times S \rightarrow \mathbb{R} \quad \forall i$$

$$\text{ii) Policy f}^n: g_c^i, \{g_{a'(s')}^i\}_{s'}, g_l^i, g_k^i \quad \forall i \text{ on same domain as value f}^n$$

$$\text{iii) Pricing f}^n: Q(s, s', \vec{A}, \vec{K}), r(\vec{A}, \vec{K}, s), w(\vec{A}, \vec{K}, s)$$

$$\begin{aligned} \text{iv) Aggregate laws of motion } & H: \tilde{A}^2 \times \tilde{K}^2 \times S \rightarrow \tilde{K}^2 \\ & G: \tilde{A}^2 \times \tilde{K}^2 \times S \rightarrow \tilde{A}^2 \end{aligned}$$

- 1) Given prices and beliefs, v_i satisfies FE and policy f's optimal
 2) Priors s.t. firms max profits:

$$w(\vec{A}, \vec{K}, s) = A(s) F_L(K_1 + K_2, g'_L(A_1, K_1, \vec{A}, \vec{K}, s) + g'_L(A_2, K_2, \vec{A}, \vec{K}, s))$$

$$r(\vec{A}, \vec{K}, s) = A(s) F_K(K_1 + K_2, g'_L(A_1, K_1, \vec{A}, \vec{K}, s) + g'_L(A_2, K_2, \vec{A}, \vec{K}, s))$$

- 3) Beliefs consistent:

$$G(\vec{A}, \vec{K}, s, s') = [g'_{d(s)}(A_1, K_1, \vec{A}, \vec{K}, s), g'_{d(s)}(A_2, K_2, \vec{A}, \vec{K}, s)] \quad \forall s'$$

$$H(\vec{A}, \vec{K}, s) = [g'_K(A_1, K_1, \vec{A}, \vec{K}, s), g'_K(A_2, K_2, \vec{A}, \vec{K}, s)]$$

- 4) Markets clear:

$$\sum_i g'_{d(s')} (A_i, K_i, \vec{A}, \vec{K}) = 0 \quad \forall s'$$

$$\begin{aligned} & \sum_i g'_L(A_i, K_i, \vec{A}, \vec{K}, s) + \sum_i g'_K(A_i, K_i, \vec{A}, \vec{K}, s) \\ &= A(s) F \left[\sum_i K_i, \sum_i g'_L(A_i, K_i, \vec{A}, \vec{K}, s) \right] \end{aligned}$$

- Could also do setup with assets that include both Arrow securities and physical capital. In this case market clearing ensures that tomorrow's physical capital is the sum of all traded assets today. We briefly discussed this in Tim's course and also in Chari's slides. Let individual assets be d and aggregate D .

$$\text{Asset market clearing: } \sum_i g_{d(s')} (D_i, \vec{D}_i, s) = \underbrace{D'}_{\text{constant } \forall s'} > 0 \quad \forall s'$$

$= D'(s')$
by consistency

$\Rightarrow$ Physical capital for tomorrow does not depend on s' ; although $D'(s')$ might.

The agent's constraints would then be:

$$c_i + \sum_{s'} Q(s, s', \vec{D}) d'_i(s') \leq w(\vec{D}, s) l_i + (1 + r(\vec{D}, s) - \delta) d_i$$

$$d'_i(s') \geq -\beta \quad \forall s'$$

$$\vec{D}'(s') = M(\vec{D}, s, s') \quad \forall s'$$

$$c_i \geq 0, \quad l_i \in [0, 1]$$

Pricing functions for wage and capital rental rates then be:

$$w(\vec{D}, s) = A(s) F_L \left(\sum_i D_i, \sum_i g_i(D_i, \vec{D}, s) \right)$$

$$r(\vec{D}, s) = A(s) F_K \left(\sum_i D_i, \sum_i g_i(D_i, \vec{D}, s) \right)$$

3) a) We want to write the following sequence form of social planners problem into a dynamic programming problem (i.e. Bellman):

$$v(c_{-1}, k_0, s_0) \equiv \max_{\{c_t(s^t), k_{t+1}(s^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t(s^t), c_{t-1}(s^{t-1})) \pi(s^t)$$

s.t. $c_t(s^t) + k_{t+1}(s^t) - (1-\delta)k_t(s^{t-1}) \leq A(s^t)F(k_t(s^{t-1}), 1)$ $\forall t, s^t$

$c_t(s^t), k_{t+1}(s^t) \geq 0 \quad \forall t, s^t$

$$c_{-1} \equiv c_{-1}(s_{-1}), s_0, k_0 \text{ given}$$

We assume s follows a first order Markov process.

Rewrite the above as:

$$\begin{aligned} & \max_{c_0(s_0), k_1(s_0)} u(c_0(s_0), c_{-1}) \\ & + \max_{\{c_t(s^t), k_{t+1}(s^t)\}_{t=1}^{\infty}} \underbrace{\sum_{t=1}^{\infty} \sum_{s^t} \beta^t u(c_t(s^t), c_{t-1}(s^{t-1})) \pi(s^t)}_{\beta \sum_{t=0}^{\infty} \sum_{s^{t+1}} \beta^t u(c_{t+1}(s^{t+1}), c_t(s^t)) \pi(s^{t+1})} \\ & \Rightarrow \beta \sum_{s_1} \left(\sum_{t=0}^{\infty} \sum_{s^{t+1}} \beta^t u(c_{t+1}(s^{t+1}), c_t(s^t)) \pi(s^{t+1} | s^t) \right) \pi(s_1 | s_0) \end{aligned}$$

$$\text{s.t. } c_0(s_0) + k_1(s_0) \leq A(s_0)F(k_0, 1)$$

$$c_t(s^t) + k_{t+1}(s^t) \leq A(s^t)F(k_t(s^t), 1) \quad \forall t \geq 1, s^t$$

non-neg

We can further rewrite this as:

$$\max_{c_0, k_1} u(c_0, c_{-1}) + \beta \sum_{s_1} \pi(s_1 | s_0) \left[\begin{array}{l} \max_{\{c, k\}} \sum_{t=0} \sum_{s^{t+1}} u(c_{t+1}(s^{t+1}), c_t(s^t)) \pi(s^{t+1} | s^t) \\ \text{s.t. } c_{t+1}(s^{t+1}) + k_{t+2}(s^{t+1}) \leq A(s^{t+1}) F(k_{t+1}(s^{t+1})) \\ \text{non-neg} \quad \quad \quad \forall t \geq 0, s^{t+1} \\ c_0, k_1, s_1 \text{ given} \end{array} \right]$$

$$\text{s.t.} \quad c_0 + k_1 \leq A(s_0) F(k_0, 1)$$

$$\text{non-neg ; } k_0, s_0, c_{-1} \text{ given}$$

We arrive at the Bellman equation:

$$v(c_{-1}, k_0, s_0) = \max_{c_0, k_1} u(c_0, c_{-1}) + \beta \sum_{s_1} v(c_0, k_1, s_1) \pi(s_1 | s_0)$$

$$\text{s.t.} \quad c_0 + k_1 \leq A(s_0) F(k_0, 1)$$

$$c_0, k_1 \geq 0$$

We can drop the time subscripts to get:

$$v(c_{-1}, k, s) = \max_{c, k'} \left\{ u(c, c_{-1}) + \beta \mathbb{E}[v(c, k', s') | s] \right\}$$

$$\text{s.t.} \quad c + k' \leq A(s) F(k, 1)$$

$$c, k' \geq 0$$

which is the social planner's problem written recursively.
The above steps were just to show how we can derive Bellman from sequence problem.

You can also see that C_{-1} must be a state variable by the following argument. If both C_0 and C_{-1} were control variables today, you choose both today. But then tomorrow you would pick C_1 and C_0 (again). So you pick C_0 in two consecutive periods. We avoid this when we choose C_{-1} as a state variable.

d) RCE : First, value f^n :

$$v(k, c_{-1}, K, C_{-1}, s)$$

$$= \max_{c, k'} \left\{ u(c, c_{-1}) + \beta E[v(k', c, K', C_{-1}, s') | s] \right\}$$

$$\text{s.t. } c + k' \leq w(K, s) + r(K, s)k$$

$$C_{-1} = H(K, C_{-1}, s)$$

$$K' = G(K, C_{-1}, s)$$

$$c, k' \geq 0$$

Policy f^n s:

$$c^* = g_c(k, c_{-1}, K, C_{-1}, s)$$

$$k'^* = g_k(k, c_{-1}, K, C_{-1}, s)$$

Need C_{-1} as a state variable since past consumption impacts utility, and thus prices. We don't want choice of c today to impact prices tomorrow, so we distinguish b/t aggregate and individual consumption $\Rightarrow$ just like Big K and little k .

Note that this is easier to see if we included Arrow securities, price of $a(s')$ is $q(C_{-1}, K, s, s')$. Since C_{-1} impacts utility today, it will likely affect intertemporal MRS, which will then affect Arrow sec. prices. So use C_{-1} to account for this.

Let $k \in X$, $c \in Y$, $s \in S$, then

A RCE is defined by:

- i) value function: $v: X \times Y \times X \times Y \times S \rightarrow \mathbb{R}$
- ii) Policy fns g_c, g_k on same domain
- iii) Prices $w(k, s), r(k, s)$
- iv) Laws of motion: H, G , such that

1) v satisfies Bellman/FE given prices and beliefs, and g_c, g_k are PFs

2) Firms maximize profits given prices:

$$w(k, s) = A(s)F_k(k, l)$$

$$r(k, s) = A(s)F_l(k, l)$$

3) Consistency of beliefs:

$$G(k, C_{-1}, s) = g_c(k, C_{-1}, k, C_{-1}, s)$$

$$H(k, C_{-1}, s) = g_k(k, C_{-1}, k, C_{-1}, s)$$

4) Markets clear:

$$\begin{aligned} g_c(k, C_{-1}, k, C_{-1}, s) + g_k(k, C_{-1}, k, C_{-1}, s) \\ = A(s)F(k, l) \end{aligned}$$

In the problems we have seen thus far, any individual state variable must have a corresponding aggregate state variable; and the union of these aggregate state variables (along w/ any agg. shock vars) is the aggregate state of the economy.

[And the variables corresponding to the aggregate state must be included as state variables for all agents.]

For example, in previous final exams, this same question was asked but with an additional

agent with "normal" preferences: $E_0 \left[\sum_{t=0}^{\infty} u_2(c_t(s^t)) \right]$.

What are the individual state variables for agent 2?

• Individual states for 2: k_2, a_2

What are individual state variables for agent 1?

• Individual state for 1: $c_{-1,1}, k_1, a_1$

Thus the aggregate state variables are:

$X \equiv (\overset{\text{Aggregate "shock"}}{\tilde{s}}, \underbrace{k_1, k_2}_{\tilde{K}}, \underbrace{a_1, a_2}_{\tilde{A}}, c_{-1,1})$. [An agg. state var for each indiv state var]

Each agents' state variables consist of their individual state variable and the aggregate state X .

In this example, we had a finite number of agents, both of measure 1.

But this idea extends to $N < \infty$ agents each of finite measure; and can also extend to when we have an infinite number of agents (i.e. a continuum) but with total finite measure. For example, this is the setting in our "Huggett" problem w/ y_t Markov. In this case, we had an infinite number of agents of total measure of one; and with individual state variables (b_i, y_i) each. Thus the aggregate state must describe the individual state (b_i, y_i) for all (an infinite number of) agents. Thus we use a distribution over all (b_i, y_i) for all agents as the aggregate state; and the prob. measure $\mathbb{P}$, which describes the distribution over bonds and income is included as the aggregate state variable.