

Adding Fiscal Policy

In 8105 and the first half of 8106, we assumed that there was no government in our models. We now add an additional agent to our model, the "government" who implements taxes to finance policy. This will allow us to investigate how households/consumers would optimally react to different types of taxes.

Adding policy to our model also enables us to derive results on what the "optimal" policy is.

• References for these notes include:

- ↳ Professor Jones' notes. I highly recommend reading.
- ↳ Chapter 16 of Ljungqvist and Sargent's "Recursive Macroeconomic Theory." This book is available online through the UMN library. This chapter covers the Ramsey Problem. Sections 16.1-16.7 are useful.
- ↳ Jonathan Heathcote's notes "The Primal Approach to Optimal Taxation"
- ↳ Past TA notes and Patrick Kiernan.

Any errors are my own.

Tax Distorted Competitive Equilibrium

We will now add fiscal policy and government to our competitive equilibria (CE).

This type of CE is called a "tax-distorted" CE (TDCE). In a TDCE we assume that the consumer(s) and firm(s) take both prices and fiscal policy as given. This is a key point for this model; if consumer(s) or firm(s) believed they could impact government policy, they would likely behave differently. So we avoid this additional complication by assuming that consumer(s) and firm(s) believe their actions do not impact fiscal policy.

We will generally assume that the government has some combination of the following tax instruments:

i) Distortionary taxes:

τ_{ct} : tax on c_t

τ_{xt} : tax on x_t

τ_{kt} : tax on $r_t^k k_t$ (capital services income)

τ_{nt} : tax on $w_t n_t$ (labor income)

ii) Lump sum transfers : T_t^i (If $T_t^i < 0$ it is a lump sum tax;
If $T_t^i > 0$ it is a lump sum payment)

We also often assume the government purchases some of the output good, which we call gov. spending, g_t .

Next, we define a TDCE with a rep. consumer and firm.

For this TDCE, we assume the consumer is taxed and faces a present value (Arrow-Debreu) budget constraint.

We also assume the government balances its

present value budget constraint. As you have seen

in PS 3, you can make different assumptions about the government's budget constraint.

Note I assume F is CRS so that profits are zero.

Definition: A TDCE in the above setup is:

i) Household quantities: $\hat{Z}^H \equiv \{\hat{c}_t, \hat{x}_t, \hat{k}_{t+1}, \hat{l}_t, \hat{n}_t\}_{t=0}^{\infty}$

ii) Firm quantities: $\hat{Z}^F \equiv \{\hat{y}_t^f, \hat{n}_t^f, \hat{k}_t^f\}_{t=0}^{\infty}$

iii) Prices: $\hat{P} \equiv \{\hat{p}_t, \hat{r}_t^k, \hat{w}_t\}_{t=0}^{\infty}$

iv) Fiscal Policy: $\hat{G} \equiv \{\tau_{ct}, \tau_{xt}, \tau_{nt}, \tau_{kt}, T_t, g_t\}_{t=0}^{\infty}$

given $k_0 > 0$, such that:

1) Given $\hat{P}$ and $\hat{G}$, the representative consumer chooses $\hat{Z}^H$ to solve:

$$\max_{\hat{Z}^H} \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

$$\text{s.t.} \quad \sum_{t=0}^{\infty} \hat{p}_t (1 + \tau_{ct}) c_t + \hat{p}_t (1 + \tau_{xt}) x_t \\ \leq \sum_{t=0}^{\infty} (1 - \tau_{nt}) \hat{w}_t n_t + (1 - \tau_{kt}) \hat{r}_t^k k_t + T_t$$

$$k_{t+1} \leq x_t + (1 - \delta) k_t \quad \forall t \geq 0$$

$$n_t + l_t \leq \bar{n}_t = 1 \quad \forall t \geq 0$$

$$c_t, k_{t+1}, n_t, l_t \geq 0 \quad \forall t \geq 0$$

k_0 given

2) Given $\hat{p}$ and $\hat{G}$, the rep firm chooses $\hat{z}^f$ to solve:

$$\begin{aligned} \max_{z^f} \quad & \sum_{t=0}^{\infty} \hat{p}_t y_t^f - \hat{r}_t k_t^f - \hat{w}_t n_t^f \\ \text{s.t.} \quad & y_t^f \leq F(k_t^f, n_t^f) \\ & y_t^f, k_t^f, n_t^f \geq 0 \end{aligned}$$

3) Markets clear $\forall t \geq 0$

$$\hat{c}_t + \hat{x}_t + \hat{g}_t = \hat{c}_t + \hat{k}_{t+1} - (1-\delta)\hat{k}_t + g_t = F(\hat{k}_t, \hat{n}_t)$$

$$\hat{n}_t^f = \hat{n}_t$$

$$\hat{k}_t^f = \hat{k}_t$$

4) Government balances its budget in present value:

$$\sum_{t=0}^{\infty} \hat{p}_t g_t + T_t = \sum_{t=0}^{\infty} \hat{p}_t \tau_c \hat{c}_t + \hat{p}_t \tau_x \hat{x}_t + \tau_w \hat{w}_t \hat{n}_t + \tau_k \hat{r}_t \hat{k}_t$$

This completes the definition. ■

Now, let's look at the optimality conditions.

Assume our general assumptions on utility function and production function. (I.e. U is st. increasing, st. concave, continuous, and differentiable; F is CRS, increasing and concave; also both satisfy Inada).

Under these conditions, there will be a unique equilibrium allocation (where $c_t, k_t, n_t, l_t > 0 \quad \forall t$).

Ignoring non-negativity, the consumer's Lagrangian is:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) + \lambda \left(\sum_{t=0}^{\infty} (1-\tau_{nt}) \hat{w}_t (1-l_t) + (1-\tau_{kt}) \hat{r}_t^k k_t + T_t - \sum_{t=0}^{\infty} \hat{p}_t [(1+\tau_{lt}) c_t + (1+\tau_{xt}) (\underbrace{k_{t+1} - (1-\delta)k_t}_{x_t})] \right)$$

FOCs :

$$[c_t]: \beta^t \underbrace{u_c(t)}_{= \frac{\partial u(c_t, l_t)}{\partial c_t}} - \lambda \hat{p}_t (1+\tau_{lt}) = 0$$

$$[n_t]: \beta^t u_l(t) - \lambda \hat{w}_t (1-\tau_{nt}) = 0$$

$$[k_{t+1}]: -\lambda \hat{p}_t (1+\tau_{xt}) + \lambda \hat{p}_{t+1} (1+\tau_{x_{t+1}}) (1-\delta) + \lambda (1-\tau_{k_{t+1}}) \hat{r}_{t+1}^k = 0$$

From the k_{t+1} FOC, we get the "Arbitrage condition"

$$\hat{p}_t (1+\tau_{xt}) = \hat{p}_{t+1} (1-\delta) (1+\tau_{x_{t+1}}) + \hat{r}_{t+1}^k (1-\tau_{k_{t+1}})$$

We also have our usual rep. firm FOCs :

$$[k_t^f]: \hat{r}_t^k = \hat{p}_t F_k(t)$$

$$[n_t^f]: \hat{w}_t = \hat{p}_t F_n(t)$$

Combining these, we get the tax distorted versions of:

$$\begin{aligned} \text{Euler: } \frac{u_c(t)}{u_c(t+1)} &= \beta \frac{(1+\tau_{ct})}{(1+\tau_{c,t+1})} \frac{\hat{p}_t}{\hat{p}_{t+1}} \\ &= \beta \frac{(1+\tau_{ct})}{(1+\tau_{c,t+1})} \left[(1-\delta) \frac{(1+\tau_{x,t+1})}{(1+\tau_{x,t})} + F_k(t+1) \frac{(1-\tau_{k,t+1})}{(1+\tau_{k,t})} \right] \end{aligned}$$

$$\text{Intra. MRS: } \frac{u_c(t)}{u_l(t)} = \frac{(1+\tau_{ct})}{(1-\tau_{nt})} \frac{\hat{p}_t}{\hat{w}_t} = \frac{(1+\tau_{ct})}{(1-\tau_{nt})} \frac{1}{F_n(t)}$$

You can see that if all distortionary taxes were set to zero, these are the same conditions as in our "normal" CEs.

Characterizing a TDCE

Given a specific fiscal policy, the following equations characterize this TDCE :

$$p_0 = 1$$

$$p_t \frac{(1 + \tau_{ct})}{(1 + \tau_{co})} = \beta^t \frac{u_c(t)}{u_c(0)} \quad \forall t$$

$$\frac{u_c(t)}{u_c(t+1)} = \beta \frac{(1 + \tau_{ct})}{(1 + \tau_{c,t+1})} \left[(1 - \delta) \frac{(1 + \tau_{x,t+1})}{(1 + \tau_{x,t})} + F_k(t+1) \frac{(1 - \tau_{k,t+1})}{(1 + \tau_{x,t})} \right] \quad \forall t$$

$$\frac{u_c(t)}{u_x(t)} = \frac{(1 + \tau_{ct})}{(1 - \tau_{nt})} \frac{1}{F_n(t)} \quad \forall t$$

k_0 given

$$\lim_{t \rightarrow \infty} p_t k_{t+1} = 0$$

$$\sum_{t=0}^{\infty} p_t g_t + T_t = \sum_{t=0}^{\infty} p_t \tau_{ct} c_t + p_t \tau_{xt} x_t + \tau_m w_t n_t + \tau_k r_t^k k_t$$

$$c_t + x_t + g_t = F(k_t, n_t) \quad \forall t$$

$$w_t = p_t F_n(t) \quad \forall t$$

$$r_t^k = p_t F_k(t) \quad \forall t$$

$$k_{t+1} \leq x_t + (1 - \delta) k_t \quad \forall t$$

Given our assumptions, these equations are necessary and sufficient for a TDCE.

Why don't we also need the consumer's budget constraint? Because we can construct it using the equations above.

Since F is CRS, we have:

$$F(k_t, n_t) = F_k(t)k_t + F_n(t)n_t$$

From the goods market clearing condition, we then get:

$$c_t + k_{t+1} - (1-\delta)k_t + g_t = \underbrace{F_k(t)k_t}_{r_t^k/p_t} + \underbrace{F_n(t)n_t}_{w_t/p_t}$$

Multiplying both sides by p_t and rearranging, we get:

$$p_t g_t = r_t^k k_t + w_t n_t - p_t c_t - p_t \underbrace{(k_{t+1} - (1-\delta)k_t)}_{\tilde{x}_t}$$

Plug in this equation for $p_t g_t$ into the government budget constraint, rearrange, and you will get the consumer's BC. If we instead used the consumer's BC in the set of characterizing equations, we could follow almost the exact same steps to obtain the government BC (this is related to Q1 PS3).

Different Tax Systems with Same Eqbm. Allocation

Consider the household's optimality conditions:

$$\frac{u_c(t)}{u_c(t+1)} = \beta \frac{(1+\tau_{ct})}{(1+\tau_{ct+1})} \left[\frac{(1-\delta)(1+\tau_{xt+1})}{(1+\tau_{xt})} + F_k(t+1) \frac{(1-\tau_{kt+1})}{(1+\tau_{kt})} \right]$$

$$\frac{u_c(t)}{u_x(t)} = \frac{(1+\tau_{ct})}{(1-\tau_{xt})} \frac{1}{F_n(t)}$$

As you have seen in PS3 Q2, it is sometimes possible for two different tax systems to result in the same equilibrium allocations.

For example, suppose we have a TDCE $\{\hat{z}^H, \hat{z}^F, \hat{G}, \hat{P}\}$ given $\{\hat{\tau}_{ct}, \hat{\tau}_{ct}, \hat{\tau}_{nt}, \hat{\tau}_{xt}\}_{t=0}^{\infty}$, where $\hat{\tau}_{ct} > 0$, $\hat{\tau}_{xt} > 0$ for at least some t , with corresponding eqbm HH allocation $\hat{z}^H$. The taxes finance $\{g_t\}_{t=0}^{\infty}$ where $g_t > 0$ for at least some t (assume $T_t = 0 \quad \forall t$).

Suppose the government now decides it only wants to tax labor income and capital services income; so the new tax system is $\{\tilde{\tau}_{ct}, \tilde{\tau}_{nt}, \tilde{\tau}_{xt} = 0, \tilde{\tau}_{kt} = 0\}_{t=0}^{\infty}$ to finance $\{g_t\}_{t=0}^{\infty}$.

The HH optimality conditions under this policy is:

$$\frac{u_c(t)}{u_c(t+1)} = \beta \left[(1-\delta) + F_k(t+1)(1-\tilde{\tau}_{k,t+1}) \right]$$

$$\frac{u_c(t)}{u_n(t)} = \frac{1}{(1-\tilde{\tau}_{nt})} \frac{1}{F_n(t)}$$

To achieve the same equilibrium allocation in the "hat" economy as in the "tilde" economy, we will require that $\{\tilde{\tau}_{nt}, \tilde{\tau}_{kt}\}_{t=0}^{\infty}$ satisfy (given $\hat{z}^H$) $\forall t$:

$$\begin{aligned} \frac{u_c(t)}{u_c(t+1)} &= \beta \frac{(1+\hat{\tau}_{ct})}{(1+\hat{\tau}_{c,t+1})} \left[(1-\delta) \frac{(1+\hat{\tau}_{x,t+1})}{(1+\hat{\tau}_{xt})} + F_k(t+1) \frac{(1-\hat{\tau}_{k,t+1})}{(1+\hat{\tau}_{kt})} \right] \\ &= \beta \left[(1-\delta) + F_k(t+1)(1-\tilde{\tau}_{k,t+1}) \right] \end{aligned}$$

$$\begin{aligned} \frac{u_c(t)}{u_n(t)} &= \frac{(1+\hat{\tau}_{ct})}{(1-\hat{\tau}_{nt})} \frac{1}{F_n(t)} \\ &= \frac{1}{(1-\tilde{\tau}_{nt})} \frac{1}{F_n(t)} \end{aligned}$$

We can rearrange the equations above to get explicit formulas for $\tilde{\tau}_{k,t+1}$ and $\tilde{\tau}_{nt}$ as functions of the "hat" taxes (and the $\hat{z}^H$ values).

So we have $\tilde{\tau}_{kt+1}$ and $\tilde{\tau}_{nt} \forall t \geq 0$, but we still need $\tilde{\tau}_{k0}$. We can pin down $\tilde{\tau}_{k0}$ by making it satisfy the consumer's BC in the "tilde" economy:

$$\sum_{t=0}^{\infty} \tilde{p}_t \hat{c}_t + \tilde{p}_t \hat{x}_t = \sum_{t=0}^{\infty} (1 - \tilde{\tau}_{nt}) \tilde{w}_t \hat{n}_t + (1 - \tilde{\tau}_{kt}) \tilde{r}_t^k \hat{k}_t$$

You can rearrange this to get a formula for $\tilde{\tau}_{k0}$.

However, we also need $\tilde{p}_t$, $\tilde{w}_t$ and $\tilde{r}_t^k$. We can obtain them from the characterizing equations from earlier:

$$\tilde{p}_0 = 1$$

$$\tilde{p}_t = \beta^t \frac{u_c(t)}{u_c(0)} = \hat{p}_t \frac{(1 + \hat{\tau}_{ct})}{(1 + \hat{\tau}_{c0})}$$

$$\tilde{r}_t^k = \tilde{p}_t F_k(t)$$

$$\tilde{w}_t = \tilde{p}_t F_n(t)$$

Following from Q1 of PS3, we also know the government budget constraint binds.

Thus if $\tilde{\tau}_{kt}$ and $\tilde{\tau}_{nt}$ satisfy the above equations for all t , then $\{\hat{z}^H, \hat{z}^F, \tilde{G}, \tilde{P}\}$ is a TDCE with fiscal policy $\tilde{G} \equiv \{\tilde{\tau}_{kt}, \tilde{\tau}_{nt}, g_t\}_{t=0}^{\infty}$.

Thus, even with different tax systems, TDCEs can have the same equilibrium allocation $\{\hat{z}^H, \hat{z}^F\}$.

As said on page 34 of Professor Jones' notes, "In this sense, $[\tau_{xt}$ and $\tau_{ct}]$ are redundant."

Thus, in the rest of the notes, we will only consider TDCEs with distortionary taxes on wage or capital services income (τ_{nt} and/or τ_{kt}).

Solving a TDCE

We have used a social planner's problem to solve for CE allocations previously. Why could we do this? Because in the economies we worked with previously, the First Welfare Theorem always held; any equilibrium allocation was PE. And since we knew any solution to a planner's problem was PE, we just used the solution to the SPP as the equilibrium allocation (if not representative consumer would use Negishi's method). But how did we know the solution to the SPP was also the equilibrium allocation? We can show that the equations that characterize the CE are identical to those that characterize the solution to the SPP.

Under our usual assumptions (listed earlier), the ADE (with unique allocation) is characterized by the following equations:

$$\text{Euler: } \frac{u_c(t)}{u_c(t+1)} = \beta \left[(1-\delta) + \overbrace{F_k(t+1)}^{= r_{t+1}^*/p_t} \right] = \beta \frac{p_t}{p_{t+1}} \quad \forall t$$

$$\text{Intra. MRS: } \frac{u_k(t)}{u_c(t)} = F_n(t) = \frac{w_t}{p_t} \quad \forall t$$

$$\text{MC: } c_t + k_{t+1} - (1-\delta)k_t = F(k_t, n_t) \quad \forall t$$

$$\text{TVC: } \lim_{t \rightarrow \infty} \beta^t u_c(t) k_{t+1} = 0 \quad \forall t$$

Initial: k_0 given

Because of the CRS production function, you can also show that these equations imply that the consumer's BC holds with equality:

$$c_t + k_{t+1} - (1-\delta)k_t = F(k_t, n_t) \stackrel{\text{CRS}}{=} F_k(t)k_t + F_n(t)n_t$$

Multiply both sides by p_t :

$$p_t c_t + p_t (k_{t+1} - (1-\delta)k_t) = \underbrace{p_t F_k(t)k_t}_{r_t^* k_t} + \underbrace{p_t F_n(t)n_t}_{w_t n_t}$$

Summing $\forall t$ will give you the ADE BC.

Now, consider our usual SPP (where $\bar{n}_t = 1 \forall t$):

$$\begin{aligned} \max_{\{c_t, k_{t+1}, n_t, l_t\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \\ \text{s.t.} \quad & c_t + k_{t+1} - (1-\delta)k_t = F(k_t, n_t) \\ & 0 \leq n_t + l_t \leq 1 \\ & c_t, k_{t+1}, n_t, l_t \geq 0 \\ & k_0 \text{ given} \end{aligned}$$

If you let λ_t be the Lagrange multiplier on the goods feasibility condition, the following equations characterize the (unique) solution to the SPP:

$$\frac{u_c(t)}{u_c(t+1)} = \beta [(1-\delta) + F_{k,t+1}] = \beta \frac{\lambda_t}{\lambda_{t+1}} \quad \forall t$$

$$\frac{u_l(t)}{u_c(t)} = F_{n,t} \quad \forall t$$

$$c_t + k_{t+1} - (1-\delta)k_t = F(k_t, n_t) \quad \forall t$$

$$\lim_{t \rightarrow \infty} \beta^t u_c(t) k_{t+1} = 0$$

k_0 given

We see that with setting $p_t = \lambda_t \quad \forall t$, the equations are identical to those in the ADE. Given the uniqueness of ADE allocation and SPP solution, we see that the ADE allocation and SPP solution coincide.

In our economies where the FWT held, we could go directly to the SPP.

However, once we add any distortionary taxes to the economy, we can no longer appeal to the FWT. See pages 19-20 in Professor Jones' notes for the welfare theorems in economies with fiscal policy. The key idea is that given $\{g_t\}_{t=0}^{\infty}$, the welfare theorems hold when only lump sum transfers are used (also need assumptions on u, F).

If distortionary taxes are used, there is no guarantee we can use a social planner's problem to solve for the eqbm allocation. However, in certain cases, we can use a SPP. We will show an example from 2011/2022 Prelim on the next several pages.

2011/2022 Prelim Setup:

↳ Discounted log utility: $\sum_{t=0}^{\infty} \beta^t \log(c_t)$

↳ Labor supplied inelastically at 1

↳ Cobb Douglas Production function: $F(k_t, n_t) = A k_t^{\alpha} n_t^{1-\alpha} = y_t$

↳ Full depreciation: $\delta = 1$

a) Guess functional form of value function and solve for policy function. $\Rightarrow$ We have done this many times.

$$k_t = (\alpha \beta A) \sum_{s=0}^{t-1} k_0^{\alpha^s}$$

$$y_t = A(\alpha \beta A) \sum_{s=0}^{t-1} k_0^{\alpha^{s+1}}$$

Now assume $\tau_{kt} = \tau_{nt} = \tau$ and uses revenue to purchase output g_t in each period.

Assume that government balances the budget in each period.

b) Set up and define a TDCE in this economy.

We did this in recitation. It is the same as the TDCE defined earlier in these notes, other than $\tau_{ct} = \tau_{xt} = \tau_t = 0 \quad \forall t$, and the government budget constraint is different:

$$\hat{p}_t g_t = \tau [\hat{w}_t \hat{n}_t + \hat{r}_t^k \hat{k}_t] \quad \forall t$$

The government budget constraint often differs across problems, so read each question carefully.

c) Show that the TDCE allocation solves a planner's problem and give this problem.

To solve these problems, first write down the equations that characterize the TDCE:

$$\frac{\hat{c}_{t+1}}{\hat{c}_t} = \beta(1-\tau)\alpha A k_{t+1}^{\alpha-1} = \beta \frac{p_t}{p_{t+1}} \quad \forall t \quad (i)$$

$$\hat{c}_t + \hat{k}_{t+1} + g_t = A \hat{k}_t^\alpha \quad \forall t \quad (ii)$$

$$\lim_{t \rightarrow \infty} \frac{\beta}{\hat{c}_t} \hat{k}_{t+1} = 0 \quad (iii)$$

$$k_0 \text{ given} \quad (iv)$$

$$g_t = \tau A \hat{k}_t^\alpha \quad (*) \quad \forall t \quad (v)$$

+ Firm FOCs $\forall t$

$$\begin{aligned} (*) \text{ From Gov BC: } p_t g_t &= \tau [\hat{w}_t n_t + \hat{r}_t^k k_t] \\ p_t g_t &= \tau [\underbrace{\hat{p}_t F_n(t)}_{=\hat{w}_t} n_t + \underbrace{\hat{p}_t F_k(t)}_{=\hat{r}_t^k} k_t] \\ \cancel{p_t} g_t &= \tau \cancel{p_t} [\underbrace{F_n(t) n_t + F_k(t) k_t}_{F(k_t, n_t) \text{ by CRS}}] \\ \Rightarrow g_t &= A k_t^\alpha \end{aligned}$$

As before, we can construct the consumer's BC using these equations and F being CRS. Try it!

If we plug in $g_t = A\hat{k}_t^\alpha$ into the goods market clearing condition, we get:

$$\hat{c}_t + \hat{k}_{t+1} + \tau A\hat{k}_t^\alpha = A\hat{k}_t^\alpha$$

$$\Rightarrow \hat{c}_t + \hat{k}_{t+1} = A\hat{k}_t^\alpha - \tau A\hat{k}_t^\alpha = (1-\tau)A\hat{k}_t^\alpha$$

the above constraint looks like a feasibility constraint in a planner's problem; since $\{g_t\}_{t=0}^\infty$ is exogenous, ^{in this problem} it is not a choice variable in the SPP and thus we don't want it to appear as such in a SPP.

Looking back at the Euler equation as well,

$$\frac{\hat{c}_{t+1}}{\hat{c}_t} = \beta(1-\tau)\alpha A\hat{k}_{t+1}^{\alpha-1}$$

we see that setting $\hat{A} = (1-\tau)A$ is a good candidate for using in the SPP.

Now, let's set up the SPP using $\hat{A} = (1-\tau)A$:

No g_t !

$$\begin{aligned} \max_{\{c_t, k_{t+1}\}_{t=0}^\infty} & \sum_{t=0}^{\infty} \beta^t \log(c_t) \\ \text{s.t.} & \quad c_t + k_{t+1} = \hat{A}k_t^\alpha \quad \forall t \\ & \quad c_t, k_{t+1} \geq 0 \quad \forall t \\ & \quad k_0 \text{ given} \end{aligned}$$

As practice take the FOCs of the above problem, which will give you the following characterizing equations of the SPP:

$$\frac{\hat{c}_{t+1}}{\hat{c}_t} = \beta \alpha \hat{A} k_{t+1}^{\alpha-1} = \beta \frac{\lambda_t}{\lambda_{t+1}} \quad \forall t \quad (\text{vi})$$

$$\hat{c}_t + k_{t+1} = \hat{A} k_t^\alpha \quad \forall t \quad (\text{vii})$$

$$\lim_{t \rightarrow \infty} \frac{\beta^t}{\hat{c}_t} k_{t+1} = 0 \quad (\text{viii})$$

k_0 given

where λ_t is the Lagrange multiplier on period t feasibility.

The Euler equations (i) and (vi) are clearly equivalent

Goods Market clearing in TDCE (ii) combined with

the formula for g_t (vi) gives us (vii).

Clearly if (iii)-(iv) are satisfied in TDCE, then (vi)-(viii) are satisfied in SPP (and vice versa).

Thus, the TDCE allocation solves this SPP.

d) In this economy, $y_t = \hat{A} (\alpha \beta \hat{A})^{\sum_{s=0}^{t-1} \alpha^{s+1}} k_0^{\alpha^{t+1}}$.

Plugging in $\hat{A} = (1-\tau)A$, it is clear that output is lower when $\tau > 0$. This completes the prelim question.

Note that there are more difficult problems of this type. See last year's final exam; the solution is on Alex Wordinger's website.

In general, you cannot assume the SPP will look exactly like our "usual" SPP, as shown in that final exam question.

To solve these problems, you need to first write down the equations that characterize the TDCE.

Then, look for ways you could redefine objects in the TDCE char. equations so that a corresponding SPP will have the same char. equations.

Using your candidate redefinition, set up the SPP.

Finally, take FOCs and write down the equations that characterize the SPP. If they are equivalent to those in the TDCE, you are done. If not, you need to try a different way to redefine things.

Ramsey Problem

This primarily follows Ljungqvist & Sargent (LS) chapter 16. The chapter is excellent, and is very thorough.

It is also based on pages 35 onward in Professor Jones' notes. You should read both of these sources.

The Ramsey problem is an optimal taxation problem. The objective of the government is "to maximize households' welfare subject to raising set revenues through distortionary taxes." (pg. 613)

To choose the optimal fiscal policy, the government must know how consumers and firms would react to any policy. From pg 613 of LS: "When designing the optimal policy, the government takes into account the competitive equilibrium reactions by consumers and firms to the tax system."

The key result we cover in this class is the Chamley-Judd result, which shows that the optimal long run tax on capital is zero ($\tau_k \rightarrow 0$ as $t \rightarrow \infty$). (Given that there is an asymptotic steady state).

Note that we restrict the tax instruments to be distortionary taxes. If we allowed the government to use lump-sum taxes to finance $\{g_t\}_{t=0}^{\infty}$, it would optimally choose to use only the lump sum taxes. However, this is not very interesting, and distortionary taxes seem closer to real life.

Related to this idea, we also restrict τ_{ko} to some number (or bound it below 1), since it essentially acts as a lump sum tax. Otherwise, the government would set τ_{ko} as high as needed to finance all spending. We also usually bound other taxes as well.

As stated on page 620 of LS, the Ramsey problem "is to choose the best competitive equilibrium" that maximizes consumer utility.

In §106, we set up the Ramsey Problem using the "Primal Approach."

In this approach, "the government can be thought of as directly choosing a feasible allocation,^④ subject to constraints that ensure the existence of prices and taxes such that the chosen allocation is consistent with the optimization behavior of households and firms." (pg. 625, LS)

Assume the rep. consumer's utility function is: $\sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$

The Ramsey problem in this approach is (assuming g_t fixed^{③③}):

$$\begin{aligned} \max_{\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}} \quad & \sum \beta^t u(c_t, l_t) \\ \text{s.t.} \quad & c_t + k_{t+1} - (1-d)k_t + g_t = F(k_t, 1-l_t) \quad \forall t \end{aligned}$$

Implementability constraint (IC)

The IC can be thought of as the "real" version of the consumer's budget constraint. To get the IC, you use the consumer and firm FOCs to substitute out the prices and taxes from the consumer's BC.

④ The dual approach is when the government chooses tax rates to maximize consumer utility.

③③ g_t can be endogenous depending on the problem.

Assume that the government uses labor and capital income taxes to finance an exogenous $\{g_t\}_{t=0}^{\infty}$.
The government balances its present value budget constraint.

$g_t > 0$ for
at least some t

The steps for the primal approach are: (adapted from LS pg. 626)

- 1) Obtain the first-order conditions of the household's and firm's problems, as well as any arbitrage pricing conditions. Solve these conditions for $\{p_t, r_t^k, w_t, \tau_{tk}, \tau_{nl}\}_{t=0}^{\infty}$ as functions of $\{c_t, l_t, k_t\}_{t=0}^{\infty}$.
- 2) Substitute these expressions for taxes and prices in terms of the allocation into the household's present value budget constraint. This is an intertemporal constraint involving only the allocation.
- 3) Solve for the Ramsey allocation by maximizing consumer utility subject to the feasibility constraint $c_t + k_{t+1} - (1-d)k_t + g_t = F(k_t, l_t)$ and the "implementability condition" derived in step 2.
- 4) After the Ramsey allocation is solved, use the formulas from step 1 to find taxes and prices.

Let's follow the steps above, which will allow us to characterize the optimal long run tax on labor income and capital.

Since the only taxes are τ_{kt} & τ_{nt} , we know the equations below characterize the TDCE:

$$p_0 = 1$$

$$p_t = \beta^t \frac{u_c(t)}{u_c(0)} \quad \forall t$$

$$\frac{u_c(t)}{u_c(t+1)} = \beta \left[(1-\delta) + F_k(t+1)(1-\tau_{kt+1}) \right] \quad \forall t \quad (\text{TDCE Euler})$$

$$\frac{u_c(t)}{u_x(t)} = \frac{1}{(1-\tau_{nt})} \frac{1}{F_n(t)} \quad \forall t \quad (\text{TDCE Intra-MRS})$$

k_0 given

$$\lim_{t \rightarrow \infty} p_t k_{t+1} = 0$$

$$\sum_{t=0}^{\infty} p_t (c_t + p_t (k_{t+1} - (1-\delta')k_t)) = \sum_{t=0}^{\infty} (1-\tau_{nt})w_t n_t + (1-\tau_{rt})r_t^k k_t$$

$$c_t + x_t + g_t = F(k_t, n_t) \quad \forall t$$

$$l_t + n_t = 1 \quad \forall t$$

$$w_t = p_t F_n(t) \quad \forall t$$

$$r_t^k = p_t F_k(t) \quad \forall t$$

We also have the arbitrage condition:

$$p_t = p_{t+1}(1-\delta') + r_{t+1}^k(1-\tau_{kt+1})$$

Following Step 1, we need formulas that for prices and taxes as a function of the allocation:

$$p_0 = 1$$

$$p_t = \beta^t \frac{u_c(t)}{u_c(0)}$$

$$(1 - \tau_{nt}) w_t = (1 - \tau_{nt}) F_n(t) p_t = p_t \frac{u_l(t)}{u_c(t)} = \beta^t \frac{u_l(t)}{u_c(0)}$$

$$w_t = p_t F_n(t) = \beta^t \frac{u_l(t)}{u_c(0)} F_n(t)$$

$$r_t^k = p_t F_k(t) = \beta^t \frac{u_k(t)}{u_c(t)} F_k(t)$$

Now, rearrange the BC:

$$\sum_{t=0}^{\infty} p_t (c_t - (1 - \tau_{nt}) w_t n_t) = \sum_{t=0}^{\infty} [(1 - \tau_{kt}) r_t^k k_t + (1 - \delta) k_t p_t - p_t k_{t+1}]$$

$$\Rightarrow RHS = \sum_{t=0}^{\infty} [(1 - \tau_{kt}) r_t^k k_t + (1 - \delta) k_t p_t] - \sum_{t=0}^{\infty} p_t k_{t+1}$$

$$= (1 - \tau_{k0}) r_0^k k_0 + (1 - \delta) p_0 k_0 + \sum_{t=1}^{\infty} [(1 - \tau_{kt}) r_t^k + (1 - \delta) p_t] k_t - \sum_{t=0}^{\infty} p_t k_{t+1}$$

$$= (1 - \tau_{k0}) r_0^k k_0 + (1 - \delta) p_0 k_0 + \sum_{t=0}^{\infty} [(1 - \tau_{k,t+1}) r_{t+1}^k + (1 - \delta) p_{t+1}] k_{t+1} - \sum_{t=0}^{\infty} p_t k_{t+1}$$

$$= [(1 - \tau_{k0}) r_0^k + (1 - \delta) p_0] k_0 + \sum_{t=0}^{\infty} \underbrace{[(1 - \tau_{k,t+1}) r_{t+1}^k + (1 - \delta) p_{t+1}] - p_t}_{=0 \text{ by arbitrage condition}} k_{t+1}$$

We need to use the TVC to ensure

$$\sum_{t=0}^{\infty} \left[(1 - \alpha_{k,t+1}) r_{t+1}^k + (1 - \delta) p_{t+1} - p_t \right] k_{t+1} \text{ converges to zero.}$$

To see this:

$$\begin{aligned} & \sum_{t=0}^{\infty} \left[(1 - \alpha_{k,t+1}) r_{t+1}^k + \underbrace{(1 - \delta) p_{t+1} - p_t}_{p_t} \right] k_{t+1} \\ &= \sum_{t=0}^{\infty} p_t k_{t+1} - p_t k_{t+1} \\ &= p_0 k_1 - p_0 k_1 + p_1 k_2 - p_1 k_2 + p_2 k_3 - p_2 k_3 + p_3 k_4 - \dots \\ &= p_0 k_1 + \lim_{T \rightarrow \infty} \sum_{t=0}^T -p_t k_{t+1} + p_{t+1} k_{t+2} \end{aligned}$$

Note that for any "T", we have a remainder term $p_{T+1} k_{T+2}$. Thus, for the sum to converge, we need $\lim_{T \rightarrow \infty} p_{T+1} k_{T+2}$ which is just the TVC (with just a change in time index).

What if, e.g. $p_t k_{t+1} = 1 \quad \forall t$?

The sum $\sum_{t=0}^{\infty} p_t k_{t+1} - p_t k_{t+1} = 1 - 1 + 1 - 1 + 1 - 1 + \dots$ is divergent.

For the budget constraint to make sense, we need $\lim_{t \rightarrow \infty} p_t = 0$. And since k_{t+1} is on a bounded set

(since the maximum sustainable exists in this problem), we know the TVC holds in the TDCE.

⊕ And we need p_t to converge to 0 "fast enough".

Returning to our BC, we now have:

$$\sum_{t=0}^{\infty} p_t [c_t - (1-\tau_{nt})w_t n_t] = [(1-\tau_{ko})r_0^k + (1-\delta)p_0]k_0$$

Now, plug in our formulas from earlier:

$$\sum_{t=0}^{\infty} \beta^t \frac{u_c(t)}{u_c(0)} c_t - \beta^t \frac{u_n(t)}{u_n(0)} n_t = [(1-\tau_{ko}) \underbrace{p_0}_{1} F_k(0) + (1-\delta) \underbrace{p_0}_{1}] k_0$$

$$\Rightarrow \sum_{t=0}^{\infty} \beta^t [u_c(t) c_t - u_n(t) n_t] = u_c(0) [(1-\tau_{ko}) F_k(0) + (1-\delta)] k_0$$

is the IC.

Thus our Ramsey problem is:

$$\max_{\{c_t, k_{t+1}, l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

$$\text{s.t. } c_t + k_{t+1} - (1-\delta)k_t + g_t = F(k_t, 1-l_t) \quad (\mu_t)$$

$$\sum_{t=0}^{\infty} \beta^t [u_c(t) c_t - u_n(t) \underbrace{l_t}_{1-l_t}] = u_c(0) [(1-\tau_{ko}) \underbrace{F_k(0)}_{\substack{\text{Fixed/} \\ \text{Bounded}}} + (1-\delta)] k_0 \quad (\lambda)$$

FOCs: For $t \geq 1$

$$[c_t]: \beta^t u_c(t) - \mu_t - \lambda \beta^t [u_c(t) + u_{cc}(t) c_t - u_{nc}(t) n_t] = 0$$

$$[l_t]: \beta^t u_n(t) - \mu_t F_n(t) - \lambda \beta^t [u_{cn}(t) c_t + u_n(t) - u_{nn}(t) n_t] = 0$$

$$[k_{t+1}]: -\mu_t + (1-\delta)\mu_{t+1} + F_k(t+1)\mu_{t+1} = 0$$

Combining these, we get the Ramsey Euler:

$$\frac{u_c(t) - \lambda [u_c(t) + u_{cc}(t)c_t - u_{xc}(t)n_t]}{u_c(t+1) - \lambda [u_c(t+1) + u_{cc}(t+1)c_{t+1} - u_{xc}(t+1)n_{t+1}]} = \beta \frac{M_t}{M_{t+1}}$$

$$= \beta [F_k(t+1) + (1-\delta)]$$

And Ramsey intra MRS:

$$\frac{u_c(t) - \lambda [u_c(t) + u_{cc}(t)c_t - u_{xc}(t)n_t]}{u_x(t) - \lambda [u_{cx}(t)c_t + u_x(t) - u_{xx}(t)n_t]} = \frac{1}{F_n(t)}$$

Now, suppose in the long run, c_t , k_{t+1} , l_t and g_t converge to their Ramsey steady state values c^{RP} , k^{RP} , l^{RP} , g^{RP} .

Then the Ramsey Euler in steady state is:

$$1 = \beta (F_k(\infty) + (1-\delta))$$

Compare this to the TDCE Euler in steady state:

$$1 = \beta ((1-\tau_{k\infty}) F_k(\infty) + (1-\delta))$$

Setting these equal, we will get that $\tau_{k\infty} = 0$.

Thus we see the Ramsey planner optimally chooses τ_{kt} to

be zero in the long-run; i.e. $\lim_{t \rightarrow \infty} \tau_{kt} = 0$.

Note that this is a limit; so τ_{kt} could be high for many years but it will eventually need to tend to 0.

How about the optimal choice of labor income taxes $\tau_{n\infty}$?

The Ramsey intra MRS in steady state is:

$$\theta^{RP} = \frac{u_c(\infty) - \lambda [u_c(\infty) + u_{cc}(\infty)c^{RP} - u_{cl}(\infty)n^{RP}]}{u_l(\infty) - \lambda [u_{cl}(\infty)c^{RP} + u_{ll}(\infty) - u_{ll}(\infty)n^{RP}]} = \frac{1}{F_n(\infty)}$$

And the TDCE intra MRS in steady state is:

$$\frac{u_c(\infty)}{u_l(\infty)} = \frac{1}{(1-\tau_{n\infty})F_n(\infty)}$$

Setting these equal, we get:

$$\theta^{RP} = \frac{u_c(\infty)}{u_l(\infty)} (1 - \tau_{n\infty})$$

$$\Leftrightarrow \tau_{n\infty} = 1 - \frac{u_l(\infty)}{u_c(\infty)} \theta^{RP}$$

Only when $\lambda = 0$ is $\tau_{n\infty} = 0$. What would $\lambda = 0$ mean? That the IC constraint is slack; so only the feasibility constraint would matter at the optimum. As Professor Jones says on page 52, $\lambda = 0$ means "using distortionary taxes has no welfare effects." In the current problem, $\lambda > 0$ almost certainly. Thus $\tau_{n\infty}$ converges to some positive number optimally.

This is the Chamley-Judd result: in the long run, do not tax capital; if you need to finance spending, tax labor income (assuming lump sum taxes not available).