

8106 Recitation 4

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1 Overview

First, we review the setup and notation for dynamic programming problems in SLP. We then walk through an exercise that is very similar to question 1 of the 2021 midterm question. We also will discuss how some proofs in SLP chapter 4 work.¹

Understanding the theory behind dynamic programming is important if you want to do quantitative work. You can write up code that solves for a value function and policy function (as you did in 8105), but how do you know your code output is actually correct? Does a solution to the Bellman equation in your research paper even exist? Should the value function be increasing? Should it be concave? What should the policy function look like? These are all things that theory can inform you about.

2 SLP's Sequence Problem (SP)

The **sequence problem (SP)** takes the following form (this notation is often called **canonical form**):

$$\begin{aligned} \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \\ \text{s.t. } x_{t+1} \in \Gamma(x_t), \quad \forall t \geq 0 \\ x_0 \in X \text{ given} \end{aligned}$$

A few notes about this problem:

- The input for this problem is x_0 ; it is “given.”
- Notice that $x_0 \in X$. Thus X is the set of values that the state variable can *possibly* take. Do not confuse this with the feasibility constraint.
- This solution $\{x_{t+1}^*\}_{t=0}^{\infty}$ to this problem is an infinite sequence.
- In period t , x_t is the state variable.²

¹Most of this material comes from SLP. Alex Wurdinger's, Conor Ryan's, and Professor Larry Jones' were also used as references for these notes. Errors are my own.

²Make sure you review the difference between state and control variables.

- In period t , x_{t+1} is the control variable that is next period's state.
- $\Gamma : X \rightarrow X$ is what I refer to as the feasibility correspondence. At period t , $\Gamma(x_t)$ describes the **set** of x_{t+1} that satisfy the feasibility constraints. Note that the set of feasible options for x_{t+1} only depends on x_t (that is, the input for Γ is x_t).
- The function $F : A \rightarrow \mathbb{R}$ is the one-period return. A is the “graph” of Γ :

$$A = \{(x_t, x_{t+1}) \in X \times X | x_{t+1} \in \Gamma(x_t)\}$$

Note that the F takes as inputs the current period state variable x_t and next period's state variable x_{t+1} .

- $\beta \geq 0$ is the stationary discount factor
- Notice that the “givens” for this problem are X , Γ , F , and β .

3 SLP's Functional Equation (FE)

The functional equation in canonical form is:

$$v(x) = \sup_{y \in \Gamma(x)} [F(x, y) + \beta v(y)], \quad \text{all } x \in X$$

Note that v is an unknown function; we are interested in finding the solution(s) to the functional equation.

A few notes about this problem:

- The input for this problem is x
- x is the current period state variable
- y is the current period control variable that is next period's state variable
- The rest of the objects are the same as in the SP, just without time subscripts

4 Principle of Optimality

Consider again the sequence problem above.

$$\begin{aligned} & \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \\ & \text{s.t. } x_{t+1} \in \Gamma(x_t), \quad \forall t \geq 0 \\ & \quad x_0 \in X \text{ given} \end{aligned}$$

Note a few properties of this problem:

- The objective function is additively separable
- The one-period return at period t , $F(x_t, x_{t+1})$, only depends on x_t and x_{t+1}

- The feasible choice at period t , $x_{t+1} \in \Gamma(x_t, x_{t+1})$, only depends on x_t and x_{t+1}

Because of the properties of this problem, we have a very neat result. As long as the sequence problem is “well-defined,” we can rewrite the problem as follows:

$$\begin{aligned}
v^*(x_0) &= \sup_{\{x_{t+1} \in \Gamma(x_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \\
&= \sup_{x_1 \in \Gamma(x_0)} F(x_0, x_1) + \sup_{\{x_{t+1} \in \Gamma(x_t)\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^t F(x_t, x_{t+1}) \\
&= \sup_{x_1 \in \Gamma(x_0)} F(x_0, x_1) + \beta \left(\sup_{\{x_{t+2} \in \Gamma(x_{t+1})\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \right) \\
&= \sup_{x_1 \in \Gamma(x_0)} F(x_0, x_1) + \beta v^*(x_1)
\end{aligned}$$

So we get:

$$v^*(x_0) = \sup_{x_1 \in \Gamma(x_0)} F(x_0, x_1) + \beta v^*(x_1)$$

It turns out that all we need to for the supremum function v^* to be a solution to the functional equation is for the SP to be well defined. Thus, if we knew the supremum function, we would also know *one solution*³ to the functional equation.

The **supremum function** $v^* : X \rightarrow \bar{\mathbb{R}} \equiv \mathbb{R} \cup \{-\infty, +\infty\}$ is defined by:

$$\begin{aligned}
v^*(x_0) &= \sup_{\{x_t\}_{t=0}^{\infty}} \lim_{n \rightarrow \infty} \sum_{t=0}^n \beta^t F(x_t, x_{t+1}) \\
&\text{s.t. } x_{t+1} \in \Gamma(x_t), \quad \forall t \geq 0 \\
&\quad x_0 \in X \text{ given}
\end{aligned}$$

This function tells us the discounted present value of utility, profits, or whatever the period return function measures, given an initial state variable x_0 . If the objective function was a lifetime utility function, the supremum function is just the *indirect utility function* of an infinitely lived agent.

Before we get to any assumptions, let's define one additional piece of notation. Let $\Pi(x_0)$ denote the set of feasible plans $\mathcal{g} \equiv \{x_t\}_{t=0}^{\infty}$ given an initial state x_0 . That is:

$$\Pi(x_0) = \{\{x_t\}_{t=0}^{\infty} : x_{t+1} \in \Gamma(x_t), \quad t = 0, 1, 2, \dots\}$$

Now we are prepared for the rest of Section 4.1. The sequence problem is well defined if the following two assumptions are satisfied:

Assumption 4.1: $\Gamma(x)$ is nonempty, for all $x \in X$. $\square$

This assumption ensures that $\Pi(x_0)$ is nonempty for all $x_0 \in X$. So there is always at

³There could be multiple functions that satisfy the functional equation.

least one feasible plan to choose from for a given x_0 .

Assumption 4.2: For all $x_0 \in X$ and $\underline{x} \in \Pi(x_0)$, $\lim_{n \rightarrow \infty} \sum_{t=0}^n \beta^t F(x_t, x_{t+1})$ exists (although it may be plus or minus infinity). $\square$

This assumption just means that the objective function attains a specific value for any feasible plan. It rules out cases where the infinite sum does not converge (like an oscillating series). For example, if $\beta = 1$ and $F(x_t, x_{t+1}) = 1$ in even periods and -1 in odd periods, the infinite sum will not converge. Thus the objective function does not attain any specific value in this case.

Assumption 4.2 is satisfied if you ensure that the period return F is bounded above **or** below, and that $\beta \in (0, 1)$. Note that we do not need F to be **both** bounded above and below for A4.2. Just one of them holding paired with $\beta \in (0, 1)$ rules out the possibility of an oscillating series like I discussed above.

If A4.1-A4.2 hold, the SP is well-defined and we can define the supremum function v^* as we did earlier. Note that v^* is a *unique* function. This is in contrast to the functional equation, which may have many solutions.

Now that we can define v^* , we are interested in its relationship to the functional equation.

Theorem 4.2: Let X , Γ , F , and β satisfy Assumptions 4.1-4.2. Then function v^* satisfies (FE). $\square$

Thus, as long as SP is well-defined, v^* is a solution to the functional equation. This is just as we had suggested after we rewrote the sequence problem at the top of page 3. However, we rarely know the exact function that defines v^* , so this is not a very helpful result at the moment. The next theorem can be more useful in certain cases.

Theorem 4.3: Let X , Γ , F , and β satisfy Assumptions 4.1-4.2. If v is a solution to (FE) and satisfies

$$\lim_{n \rightarrow \infty} \beta^n v(x_n) = 0, \text{ all } \underline{x} \in \Pi(x_0) \text{ all } x_0 \in X$$

then $v = v^*$. $\square$

In some cases where we use “guess and verify” to find a solution v to the functional equation, this theorem is helpful. As long as this “guess and verify” solution v satisfies $\lim_{n \rightarrow \infty} \beta^n v(x_n) = 0$ (the boundedness condition), we know that it is identical to v^* . Although we haven’t gone through an example in class where we could use this theorem, such examples do exist (think about examples where v is bounded above and below).

Since v^* is a unique function, Theorem 4.3 shows us that the functional equation can have at most one solution v that satisfies $\lim_{n \rightarrow \infty} \beta^n v(x_n) = 0$. If two different functions v and w were solutions to the functional equation and both satisfied the boundedness condition, then Theorem 4.3 tells us $v = v^*$ and $w = v^*$. But this implies $v = w$, which contradicts our assumption that they were different functions.

I will add the some more material to this section if i have time.

5 Single-Sector Growth Problem - Midterm 2021 (Q1)

The problem below is identical to question 1 in midterm 2021, other than in this example we assume labor does not enter the utility function and is supplied inelastically at 1. Even though this is just a single example, it will help you understand the steps you should follow when you are working with a different dynamic programming problem.

Consider the following single sector growth problem:⁴

$$\begin{aligned} \max_{c_t, k_{t+1}} \quad & \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t.} \quad & c_t + x_t \leq f(k_t) \\ & k_{t+1} \leq (1 - \delta)k_t + x_t \\ & c_t, k_{t+1} \geq 0 \\ & k_0 \text{ given} \end{aligned}$$

a.) **Write down the Bellman equation for this problem.**

First, notice that k is the state variable and c , x , and k' are the control variables. In the current period, k is predetermined while c , x , and k' can be “controlled.” Note that k' is *next* period’s state variable (even though it is a current period control variable).

Given that we do not know anything about the properties of this problem, we should write the Bellman equation in one of two ways. Either as:

$$\begin{aligned} v(k) = \max_{c, x, k'} \quad & u(c) + \beta v(k') \\ \text{s.t.} \quad & c + x \leq f(k) \\ & k' \leq (1 - \delta)k + x \\ & c, k' \geq 0 \end{aligned}$$

Or we could combine the two constraints $c + x \leq f(k)$ and $k' \leq (1 - \delta)k + x$ (to eliminate the choice of x from the problem) and write the Bellman equation as:

$$\begin{aligned} v(k) = \max_{c, k'} \quad & u(c) + \beta v(k') \\ \text{s.t.} \quad & c + k' - (1 - \delta)k \leq f(k) \\ & c, k' \geq 0 \end{aligned}$$

⁴Note that we do not have a non-negativity constraint on x , which simplifies the problem. If we did assume $x \geq 0$, this would prevent the agent from consuming out if its capital stock (I was incorrect about this in recitation). Unless I specifically ask to in the problem, do not write down the $x_t \geq 0$ constraint.

Note that we did not plug in the feasibility constraint $f(k) + (1 - \delta)k - k'$ for c in the utility function, as we do not know for sure if that constraint binds. Since we did not assume that the utility function is strictly increasing, we should not necessarily assume this.⁵

What if the question asked us to write the Bellman equation or sequence problem in SLP's canonical form? We would need to find the one period return function $F(k, k')$ and the feasibility correspondence $\Gamma(k)$.

Let's use the version of the Bellman equation where we avoid explicitly choosing x . As we have done before,⁶ we define the period return as follows:⁷

$$\begin{aligned} F(k, k') &= \max_c u(c) \\ \text{s.t. } c + k' - (1 - \delta)k &\leq f(k) \\ c &\geq 0 \\ k, k' &\text{ fixed} \end{aligned}$$

Notice that k and k' (the current state variable and next period's state variable, respectively) are treated as givens and that the problem is maximizing over c (the control variable that is **not** next period's state variable).

Remember that $\Gamma(k)$ should tell us the *set* of feasible values of k' given k . We have the constraints:

$$\begin{aligned} c + k' - (1 - \delta)k &\leq f(k) \\ c, k' &\geq 0 \end{aligned}$$

Which can be rearranged to:

$$\begin{aligned} 0 \leq k' &\leq f(k) + (1 - \delta)k - c \\ c &\geq 0 \end{aligned}$$

Clearly the maximum value k' can attain is $f(k) + (1 - \delta)k$, which occurs when $c = 0$. The minimum value of k' is also clearly zero. Thus we can write the feasibility correspondence as:⁸

$$\Gamma(k) = \{k' \in K \mid k' \in [0, f(k) + (1 - \delta)k]\}$$

Now we have the pieces to write the Bellman/functional equation in canonical form:

$$v(k) = \max_{k' \in \Gamma(k)} [F(k, k') + \beta v(k')]$$

⁵For example, what if $u(c) = 0$ for all c ? This is a completely uninteresting example, but in this case any feasible combination of c , x , and k' would be optimal to choose, even if some of the available resources were wasted. In other words, the feasibility constraint would not be required to bind at the optimum.

⁶An earlier version of these notes accidentally had $k' \geq 0$ as an additional constraint. But as $k' \in K$ is fixed, this does not make much sense to include.

⁷Since we know nothing about u or f at this point, we should probably use "sup" instead of "max" here. We are also implicitly assuming that the maximization problem is well defined (that a maximum or supremum actually exists for all feasible pairs of k and k').

⁸Note that we need a couple regularity conditions here (like strictly increasing utility function), but they are fairly harmless so we will ignore them for now.

And we can also write the sequence problem in canonical form:

$$\begin{aligned} \sup_{\{k_{t+1}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t F(k_t, k_{t+1}) \\ \text{s.t.} \quad & k_{t+1} \in \Gamma(k_t), \quad \forall t \geq 0 \\ & k_0 \in K \text{ given} \end{aligned}$$

Defining F and Γ this way is extremely general, which is very useful when we have more complex problems. For example, if leisure was included in the utility function and labor was an input to the production function, we would solve for the period return function $F(k, k')$ by fixing k and k' , and maximizing over c , n , and l (try this!).

Why does SLP use canonical form? This notation allowed the authors to focus solely on the variables that make the problem dynamic. The authors want $F(k, k')$ to tell us the maximum value of current period utility for any feasible combination of today's state and tomorrow's state; the optimal choice of the other control variables is already accounted for in the value of $F(k, k')$. The idea is similar for having $k' \in \Gamma(k)$ be the only constraint; today's state variable is the only information the agent needs to choose tomorrow's state variable.

- b.) **Provide assumptions on u , f , β , and δ that are sufficient to guarantee that there is a unique bounded and continuous function satisfying the functional equation in (b). Explain briefly why each assumption is required.**

This question asks us to write down conditions such that Theorem 4.6 of SLP is satisfied.

Theorem 4.6: Let X , Γ , F , and β satisfy Assumptions 4.3 and 4.4, and let $C(X)$ be the space of bounded continuous functions $f : X \rightarrow \mathbb{R}$, with the sup norm. then the operator T maps $C(X)$ into itself, $T : C(X) \rightarrow C(X)$; T has a unique fixed point $v \in C(X)$; and for all $v_0 \in C(X)$,

$$\|T^n v_0 - v\| \leq \beta^n \|v_0 - v\|, \quad n = 0, 1, 2, \dots$$

Moreover, given v , the optimal policy correspondence $G : X \rightarrow X$ defined by:

$$G(x) = \{y \in \Gamma(x) : v(x) = F(x, y) + \beta v(y)\}$$

is compact valued and u.h.c. $\square$

This theorem states that under A4.3 and A4.4 the operator T defined by $(Tf)(x) = \max_{y \in \Gamma(x)} [F(x, y) + \beta f(y)]$ maps from the space of continuous bounded functions $C(X)$ to the space of continuous bounded functions $C(X)$ (i.e. $T : C(X) \rightarrow C(X)$) and that there is a unique fixed point in $C(X)$ such that $T^n v_0$ converges to for any $v_0 \in C(X)$. Clearly we need the Contraction Mapping Theorem to hold for this theorem to work.

Now, let's see what we need to satisfy Assumptions 4.3 and 4.4.

Assumption 4.3: X is a convex subset of $\mathbb{R}^l$, and the correspondence $\Gamma : X \rightarrow X$

is nonempty, compact-valued, and continuous. $\square$

Assumption 4.4: The function $F : A \rightarrow \mathbb{R}$ is bounded and continuous, and $0 < \beta < 1$. $\square$

Now, let's add some assumptions to our current problem:⁹

- (a) Preferences::
 - (Ui) u is continuous¹⁰
 - (Uii) u is strictly increasing
 - (Uiii) $\beta \in (0, 1)$
- (b) Production Function f and δ :
 - (Pi) $f(0) = 0$
 - (Pii) f is continuous
 - (Piii) f is strictly increasing
 - (Piv) f is concave
 - (Pv) Inada conditions hold for f :
 - $\lim_{k \rightarrow 0} f_k(k) = +\infty$
 - $\lim_{k \rightarrow +\infty} f_k(k) = 0$
 - (Pvi) $\delta \in (0, 1]$

If you were answering this question on the exam, you would not need to go into nearly as much as detail as we go into below. However, you would need to briefly explain why the assumptions are required.

First, note that assumptions (Pi)-(Pvi) ensure that there exists a unique maximum sustainable capital stock. Remember that the maximum sustainable capital stock is a fixed point $\tilde{k} = h(\tilde{k}) = f(\tilde{k}) + (1 - \delta)\tilde{k}$. As shown in SLP, it also must satisfy the following:

- (a) $k \leq f(k) + (1 - \delta)k \leq \tilde{k}$ for all $k \in [0, \tilde{k}]$
- (b) $f(k) + (1 - \delta)k \leq k$ for all $k > \tilde{k}$

We will now argue that under these assumptions, $\tilde{k}$ exists. You could also use a chart to argue that it exists, like did in 8105.

Rearranging the formula above, we get: $\delta = \frac{f(\tilde{k})}{\tilde{k}}$. Since $\delta \in (0, 1]$, we can use the intermediate value theorem to argue that the $\tilde{k}$ exists.¹¹ Using L'Hopital's rule and our assumption about the Inada conditions, we can show that $\lim_{k \rightarrow 0} \frac{f(k)}{k} = +\infty$ and $\lim_{k \rightarrow \infty} \frac{f(k)}{k} = 0$. Since $f(k)$ is continuous on the interior of K , then so will $\frac{f(k)}{k}$. Applying the intermediate value theorem, we know there exists a $\tilde{k}$ such that $\delta = \frac{f(\tilde{k})}{\tilde{k}}$. We use the concavity of f along with the Inada conditions to ensure that there is a unique

⁹Throughout this exercise, I am not claiming that these are the *minimal* assumptions required for each step. You could use fewer assumptions but it makes verifying the SLP theorems more complicated.

¹⁰In class I said we also needed the utility function to be bounded below. However, we technically don't need it. A continuous function on a closed and bounded set is bounded above and below.

¹¹Note that if $\delta = 0$, then a finite maximum sustainable capital does not exist.

$\tilde{k}$ and that it satisfies (a) and (b) above.

Now that we have found the maximum sustainable capital, we can be explicit about our state space K . We set $K = [0, \tilde{k}]$,¹² which is clearly a convex subset of $\mathbb{R}$ (as required by Assumption 4.3).

Now, define the feasibility correspondence $\Gamma : K \rightarrow K$ by $\Gamma(k) = \{k' \in K \mid k' \in [0, f(k) + (1 - \delta)k]\}$. Since $f(0) = 0$, f is strictly increasing, and $\delta \in (0, 1]$ we know that Γ is nonempty-valued.¹³ It is also clearly compact-valued ($\Gamma(k)$ is closed and bounded $\forall k$). The continuity of Γ follows from Exercise 3.13 in SLP (which holds since $f(k) + (1 - \delta)k$ is continuous).

So we have shown that Assumption 4.3 holds under our current assumptions. Now let's show that Assumption 4.4 holds.

As above, we define the period return function $F : A \rightarrow \mathbb{R}$ as follows:

$$\begin{aligned} F(k, k') &= \max_c u(c) \\ \text{s.t. } &c + k' - (1 - \delta)k \leq f(k) \\ &c, k' \geq 0 \end{aligned}$$

To ensure that F is continuous, we can refer to the Theorem of the Maximum.

Theorem of the Maximum: Let $X \subseteq \mathbb{R}^l$ and $Y \subseteq \mathbb{R}^m$, let $f : X \times Y \rightarrow \mathbb{R}$ be a continuous function, and let $\Gamma : X \rightarrow Y$ be a compact-valued and continuous correspondence. The the function $h : X \rightarrow \mathbb{R}$ defined by

$$h(x) = \max_{y \in \Gamma(x)} f(x, y)$$

is continuous, and the correspondence $G : X \rightarrow Y$ defined by

$$G(x) = \{y \in \Gamma(x) : f(x, y) = h(x)\}$$

is nonempty, compact-valued, and u.h.c. $\square$

Pairing our assumptions on the production function, our definition of K , with our assumption that u is continuous and strictly increasing, we get:

$$F(k, k') = u(f(k) + (1 - \delta)k - k')$$

By the Theorem of the Maximum, $F(k, k')$ is continuous. F is also bounded since u is bounded over the state space. Also note that F is strictly increasing in k and strictly decreasing in k' .

We see that that under the set of assumptions above, F is bounded and continuous.

¹²Our, to be more careful, we could use $K = [0, \max\{\tilde{k}, k_0\}]$ as we did in 8105.

¹³What if $f(0) = -1$? Then when $k = 0$, there is no feasible choice for k' (the set $[0, f(0) + (1 - \delta) \cdot 0] \equiv [0, -1]$ is empty), so $\Gamma(0)$ would be empty.

All that we need left to satisfy Assumption 4.4 is for $\beta \in (0, 1)$, which we already assumed (assumption (Uiii)).

Now we have the objects we need to write the Bellman/functional equation (which is exactly the same we in (a)):

$$v(k) = \max_{k' \in \Gamma(k)} [F(k, k') + \beta v(k')]$$

Since A4.3 and A4.4 hold in our current problem, Theorem 4.6 is satisfied.

The proof of Theorem 4.6 utilizes the Theorem of the Maximum to show that the operator T defined by $(Tf)(x) = \max_{y \in \Gamma(x)} [F(x, y) + \beta f(y)]$ is continuous (so that $T : C(X) \rightarrow C(X)$). We can then show that T satisfies Blackwell's sufficient conditions¹⁴ and is a contraction; and therefore apply the Contraction Mapping Theorem. Thus there exists a **unique, bounded, continuous** function v that satisfies the functional equation above.

There is a good discussion about the implications of this theorem at the top of page 80 of SLP. Since the solution v to the functional equation is bounded, we know that it satisfies the boundedness condition from Theorem 4.3. Thus, according to that theorem, $v = v^*$. Since v is continuous and bounded, so is the supremum function. Because of the boundedness of v^* , we can also use Theorem 4.5. Suppose we solve for the v and its corresponding policy function on the computer (under the current assumptions). This result tells us that any infinite plan created by the policy function will be an optimal plan in the sequence problem. This is exactly what we want when we map the social planner's problem to a dynamic programming problem.

- c.) **Make appropriate assumptions on u , f , β , and δ to characterize the optimal policy function through the first order and envelope condition and derive the conditions.**

To use the first order and envelope conditions, we need to ensure that u , f , and v are differentiable. This points us to Theorem 4.11 of SLP.

Theorem 4.11 (Differentiability of the value function): Let X , Γ , F , and β satisfy Assumptions 4.3-4.4 and 4.7-4.9, and let v and g satisfy $v(x) = \max_{y \in \Gamma(x)} [F(x, y) + \beta v(y)]$ and $G(x) = \{y \in \Gamma(x) : v(x) = F(x, y) + \beta v(y)\}$. If $x_0 \in \text{int}X$ and $g(x_0) \in \text{int}\Gamma(x_0)$, then v is continuously differentiable at x_0 , with derivatives given by

$$v_i(x_0) = F_i[x_0, g(x_0)], \quad i = 1, 2, \dots, l \quad \square$$

Since we already showed Assumptions 4.3 and 4.4 are satisfied, we just need additional conditions on u and f to ensure that A4.7-4.9 are satisfied. After adding these conditions, any solution to the functional equation will be differentiable. Thus we can use the first order conditions with respect to k' and envelope condition. First, let's state A4.7-A4.9:

¹⁴Make sure to review Blackwell's sufficient conditions for a contraction.

Assumption 4.7: F is strictly concave; that is:

$$F[\theta(x, y) + (1 - \theta)(x', y')] \geq \theta F(x, y) + (1 - \theta)F(x', y')$$

$$\text{all } (x, y), (x', y') \in A, \text{ and all } \theta \in (0, 1)$$

and the inequality is strict if $x \neq x'$. $\square$

Assumption 4.8: Γ is convex in the sense that for any $0 \leq \theta \leq 1$, and $x, x' \in X$,

$$y \in \Gamma(x) \text{ and } y' \in \Gamma(x') \text{ implies}$$

$$\theta y + (1 - \theta)y' \in \Gamma[\theta x + (1 - \theta)x'] \quad \square$$

Assumption 4.9: F is continuously differentiable on the interior of A . $\square$

To satisfy A4.7-4.9, we just need to add a couple more assumptions:

(a.) Preferences:

(Uiv) u is strictly concave

(Uiiv) u is continuously differentiable

(b.) Production function:

(Pvii) f is continuously differentiable

Let's verify that these conditions satisfy A4.7-A4.9. To satisfy Assumption 4.7, we only need one additional condition on the utility function: u has to be strictly concave in c . The proof of this is below:

Pick any $(k_1, k'_1), (k_2, k'_2) \in A$ and any $\theta \in (0, 1)$.

$$F[\theta(k_1, k'_1) + (1 - \theta)(k_2, k'_2)]$$

$$= u[f(\theta k_1 + (1 - \theta)k_2) + (1 - \delta)(\theta k_1 + (1 - \theta)k_2) - (\theta k'_1 + (1 - \theta)k'_2)]$$

Since f is concave and since u is increasing in c , we know that the following inequality holds:

$$\geq u[\theta f(k_1) + (1 - \theta)f(k_2) + \theta(1 - \delta)k_1 + (1 - \theta)(1 - \delta)k_2 - (\theta k'_1 + (1 - \theta)k'_2)]$$

And we can rearrange the above as:

$$= u[\theta(f(k_1) + (1 - \delta)k_1 - k'_1) + (1 - \theta)(f(k_2) + (1 - \delta)k_2 - k'_2)]$$

Using that $c_1 = f(k_1) + (1 - \delta)k_1 - k'_1$ and $c_2 = f(k_2) + (1 - \delta)k_2 - k'_2$ and the strict concavity of u in c , we see:

$$\geq \theta u[f(k_1) + (1 - \delta)k_1 - k'_1] + (1 - \theta)u[f(k_2) + (1 - \delta)k_2 - k'_2] = \theta F(k_1, k'_1) + (1 - \theta)F(k_2, k'_2)$$

and the inequality strict if $k_1 \neq k_2$. $\square$

Given our current assumptions on u and f , Assumption 4.8 is satisfied as well. Verification is straightforward and primarily uses the concavity of f . The proof follows:

Take any $\theta \in [0, 1]$ and $k_1, k_2 \in K$. Let $k'_1 \in \Gamma(k_1)$ and $k'_2 \in \Gamma(k_2)$. Then:

$$\Gamma[\theta k_1 + (1 - \theta)k_2] \equiv \{k_3 \in K \mid k_3 \in [0, f(\theta k_1 + (1 - \theta)k_2) + (1 - \delta)(\theta k_1 + (1 - \theta)k_2)]\}$$

Note that since $k'_1 \in \Gamma(k_1)$ and $k'_2 \in \Gamma(k_2)$, we have:

$$0 \leq k'_1 \leq f(k_1) + (1 - \delta)k_1$$

$$0 \leq k'_2 \leq f(k_2) + (1 - \delta)k_2$$

Multiplying the first inequality by θ , the second by $(1 - \theta)$, and then adding them together we get:

$$0 \leq \theta k'_1 + (1 - \theta)k'_2 \leq \theta f(k_1) + (1 - \theta)f(k_2) + (1 - \delta)(\theta k_1 + (1 - \theta)k_2)$$

Since f is concave, we have:

$$\theta f(k_1) + (1 - \theta)f(k_2) \leq f(\theta k_1 + (1 - \theta)k_2)$$

Combining this result with the previous inequality, we get:¹⁵

$$\begin{aligned} 0 \leq \theta k'_1 + (1 - \theta)k'_2 &\leq f(\theta k_1 + (1 - \theta)k_2) + (1 - \delta)(\theta k_1 + (1 - \theta)k_2) \\ \implies \theta k'_1 + (1 - \theta)k'_2 &\in \Gamma(\theta k_1 + (1 - \theta)k_2) \quad \square \end{aligned}$$

Given that A4.3-A4.4 and A4.7-A4.8 hold, we get an intermediate result that is required for the differentiability of the value function: the value function v is strictly concave and the policy correspondence G is a continuous, single-valued function (it is a policy *function*,¹⁶ which we will denote by g). This follows directly from Theorem 4.8.

Theorem 4.8: Let X , Γ , and β satisfy Assumptions 4.3-4.4 and 4.7-4.8; let v satisfy $v(x) = \max_{y \in \Gamma(x)} [F(x, y) + \beta v(y)]$ and G satisfy $G(x) = \{y \in \Gamma(x) : v(x) = F(x, y) + \beta v(y)\}$. Then v is strictly concave and G is a continuous, single-valued function.

The proof of this theorem applies Corollary 1 to the Contraction Mapping Theorem from section 3.2.

Theorem 3.2 (Corollary 1 to the CMT): Let (S, ρ) be a complete metric space and let $T : S \rightarrow S$ be a contraction mapping with fixed point $v \in S$. If S' is a closed subset of S and $T(S') \subseteq S'$, then it follows that $v = Tv \in S'$. If in addition $T(S') \subseteq S'' \subseteq S'$, then $v \in S''$. $\square$

In the proof of Theorem 4.8, we have the following sets:

- $C(X)$: set of bounded, continuous functions on X
- $C'(X)$: set of bounded, continuous, and weakly concave functions on X
- $C''(X)$: set of bounded, continuous, and strictly concave functions on X

¹⁵Also note that $\theta k_1 + (1 - \theta)k_2 \in K$ since K is convex (A4.3).

¹⁶Make sure you understand the distinction between a correspondence and a function.

Clearly $C''(X) \subset C'(X) \subset C(X)$. From Theorem 4.6, we know that $C(X)$ is a complete metric space and that the operator $T : C(X) \rightarrow C(X)$ (defined previously) is a contraction. Since $C'(X)$ is a closed subset of $C(X)$, we just need to show that $T[C'(X)] \subseteq C''(X)$ to apply Theorem 3.2. What does $T[C'(X)] \subseteq C''(X)$ mean? It says that if the function f is bounded, continuous, and weakly concave ($f \in C'(X)$), then the function resulting from applying the operator to f (i.e. Tf) is bounded, continuous, and *strictly* concave ($Tf \in C''(X)$).

We will now show that $T[C'(X)] \subseteq C''(X)$.

Let f be a bounded, continuous, and weakly concave function ($f \in C'(X)$). Take $\theta \in (0, 1)$ and $k_1 \neq k_2 \in K$. Suppose $k'_1 \in \Gamma(k_1)$ and $k'_2 \in \Gamma(k_2)$ are the optimal choices for $(Tf)(k_1)$ and $(Tf)(k_2)$, respectively. Note that $\theta k'_1 + (1 - \theta)k'_2 \in \Gamma(\theta k_1 + (1 - \theta)k_2)$ follows from Assumption 4.8. From the definition of Tf , we have:

$$(Tf)(\theta k_1 + (1 - \theta)k_2) = \max_{y \in \Gamma(\theta k_1 + (1 - \theta)k_2)} [F(\theta k_1 + (1 - \theta)k_2, y) + \beta v(y)]$$

By the definition of a maximum, we have:

$$\geq F(\theta k_1 + (1 - \theta)k_2, \theta k'_1 + (1 - \theta)k'_2) + \beta f(\theta k'_1 + (1 - \theta)k'_2)$$

Since F is strictly concave and f is weakly concave, we have:

$$> \theta F(k_1, k'_1) + (1 - \theta)F(k_2, k'_2) + \theta \beta f(k'_1) + (1 - \theta)\beta f(k'_2)$$

Rearranging, we get:

$$\begin{aligned} &= \theta[F(k_1, k'_1) + \beta f(k'_1)] + (1 - \theta)[F(k_2, k'_2) + \beta f(k'_2)] \\ &= \theta(Tf)(k_1) + (1 - \theta)(Tf)(k_2) \end{aligned}$$

Thus we see that Tf is strictly concave:

$$(Tf)(\theta k_1 + (1 - \theta)k_2) > \theta(Tf)(k_1) + (1 - \theta)(Tf)(k_2)$$

Since $f \in C'(X)$ was arbitrary, Tf is strictly concave for all $f \in C'(X)$. Therefore $T[C'(X)] \subseteq C''(X)$. $\square$

Thus the unique fixed point, $v = Tv$, is a bounded, continuous, and strictly concave function ($v \in C''(X)$). Since v is the unique fixed point in the set of continuous, bounded functions $C(X)$, we can choose any bounded, continuous function as the initial guess v_0 ; by the Contraction Mapping Theorem $T^n v_0$ will converge to v for any $v_0 \in C(X)$.

You can get an idea of how this proof works by thinking about the nesting of the aforementioned sets and how the operator T maps between them. A good exercise is to use circles to represent the sets $C(X)$, $C'(X)$, $C''(X)$, and $T[C'(X)]$ and use an initial guess $v_0 \in C'(X)$ to show how the operator $T^n v_0$ maps between the sets. Where will the fixed point be?

By assuming that u and f are also continuously differentiable,¹⁷ we know that Assumption 4.9 is satisfied.

As A4.3-4.4 and A4.7-4.9 are satisfied, we know that Theorem 4.11 holds. Thus the unique bounded continuous solution v to the functional equation is continuously differentiable. So we can obtain the first order condition (assuming interior solutions):

$$\begin{aligned} F_{k'}(k, k') + \beta v'(k') &= 0 \\ \iff u'(f(k) + (1 - \delta)k - k') &= \beta v'(k') \end{aligned}$$

And the envelope condition:

$$\begin{aligned} v'(k) &= F_k(k, k') \\ \iff v'(k) &= u'(f(k) + (1 - \delta)k - k')(1 - \delta + f_k(k)) \end{aligned}$$

Combining these, we get the Euler equation:

$$u'(f(k) + (1 - \delta)k - k') = \beta u'(f(k') + (1 - \delta)k' - k'')(1 - \delta + f_k(k'))$$

- d.) **Use the conditions in (c) to show that the optimal policy function is strictly increasing. Which additional assumptions do you make on u , f , β , and δ ?**

We do not need any additional assumptions on u , f , β , and δ . To show that the optimal policy function g is strictly increasing in k , we will use a proof by contradiction.

Suppose that there exists $k_1, k_2 \in K$ such that $k_1 > k_2$ but $g(k_1) \leq g(k_2)$. Since f is strictly increasing, we have:

$$f(k_1) + (1 - \delta)k_1 - g(k_1) > f(k_2) + (1 - \delta)k_2 - g(k_2)$$

Since u strictly concave and continuously differentiable, we know that u' is strictly decreasing. Thus:

$$u'(f(k_1) + (1 - \delta)k_1 - g(k_1)) < u'(f(k_2) + (1 - \delta)k_2 - g(k_2))$$

Using the first order condition, this implies:

$$\beta v'(g(k_1)) < \beta v'(g(k_2))$$

Since v is strictly concave (following Theorem 4.8) and continuously differentiable (Theorem 4.11), we know that v' is strictly decreasing. So we get:

$$g(k_1) > g(k_2)$$

Which is a contradiction, since we assumed $g(k_1) \leq g(k_2)$.

Thus g is strictly increasing. $\square$

- e.) **Define a steady state for this problem. Use the conditions in (c) to show that there is a unique positive steady-state. Clearly state if you make additional assumptions on u , f , β , and δ .**

You can refer to Alex's Wurdinger's "Recitation 3 (DP and Steady State)" notes for this question. He does a good job of answering it.

¹⁷I don't think you necessarily need f to be *continuously* differentiable; just normal differentiability should work.