

Recitation 2

These notes cover some concepts from Section 4 of Professor Jones' notes. The results I shared from MWG will not be covered in the exam, so I do not include them here.

We will discuss aggregation methods for the household side^{*}. We covered two methods to reduce the model to a "representative consumer" problem:

- i) Identical consumers (identical preferences and endowments)
- ii) Identical homothetic preferences but heterogeneous endowments.

On the next page, we cover the "Identical consumers" method.

^{*} You should study firm-side aggregation as well.

Method 1: Identical Consumers

Required Assumptions

- Preferences : There are $N \in [1, \infty)$ consumers.

$$\text{Let } U^i(\{c_t^i, l_t^i\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta_i^t u_i(c_t^i, l_t^i)$$

$$\text{such that } \beta_i = \beta_j = \beta \quad \forall i, j$$

(i.e. all consumers' discount factors are same)

and $u_i = u_j = u \quad \forall i, j$ (i.e. all consumers' period utility is the same).

Thus $U^i = U^j = U \quad \forall i, j$, so that

all consumers' preferences are identical.

Also assume U^i is strictly increasing, continuous, and strictly concave (we need strict concavity for uniqueness of the optimal $\{c_t^i, l_t^i, n_t^i\}_{t=0}^{\infty}$ quantities).

[continued next page]

- Endowments: $k_0^i = k_0^j = k_0^H \quad \forall i, j$ (initial capital stock same $\forall i$)

$$\bar{n}_t^i = \bar{n}_t^j = \bar{n}_t^H \quad \forall i, j, \quad \forall t \quad (\text{time endowment same } \forall i, \forall t)$$

So all consumers have identical endowments.

- Firm: Firms across and within all sectors are identical; they have the same CRS and concave production technologies.

This means we can use the firm aggregation result and use a single representative firm in the competitive equilibrium.

End of assumptions.

For the next section, let $I \equiv \{1, 2, \dots, N\}$

denote the set of indexes for each consumer.

An ADE in this economy is an allocation-price pair:

$$\{ \underbrace{\hat{z}^H}_{\text{Allocation}}, \underbrace{\hat{z}^F}_{\text{Prices}}, \underbrace{\hat{P}}_{\text{Prices}} \}, \text{ with:}$$

• Household Quantities: $\hat{z}^H \equiv \{ \hat{z}^{Hi} \}_{i \in I}$ and

$$\hat{z}^{Hi} \equiv \{ \hat{c}_t^i, \hat{l}_t^i, \hat{n}_t^i, \hat{k}_t^i, \hat{x}_t^i \}_{t=0}^{\infty}$$

• Rep. Firm Quantities: $\hat{z}^F \equiv \{ \hat{y}_t^f, \hat{k}_t^f, \hat{n}_t^f \}_{t=0}^{\infty}$

• Prices: $\hat{P} \equiv \{ \hat{p}_t, \hat{r}_t^k, \hat{w}_t \}_{t=0}^{\infty}$

and with $k_0^i \forall i \in I$, and $\bar{n}_t^i \forall i, \forall t$ given such that:

1) Given $\hat{P}$, each consumer $i \in I$ chooses $\hat{z}^{Hi}$ to solve:

$$\begin{aligned} \max_{\hat{z}^{Hi}} \quad & \sum_{t=0}^{\infty} \beta^t u(c_t^i, l_t^i) \\ \text{s.t.} \quad & \sum_{t=0}^{\infty} \hat{p}_t (c_t^i + x_t^i) \leq \sum_{t=0}^{\infty} \hat{r}_t^k k_t^i + \hat{w}_t \hat{n}_t \end{aligned}$$

$$k_{t+1}^i \leq k_t^i + x_t^i \quad \forall t \geq 0$$

$$l_t^i + n_t^i \leq \bar{n}_t^i \quad \forall t \geq 0$$

$$c_t^i, k_{t+1}^i, n_t^i, l_t^i \geq 0 \quad \forall t$$

k_0^i given

Note: Ignoring we will ignore $x_t \geq 0$, but it shouldn't change the result

2) Given P , firm chooses $\hat{z}^f$ to solve

$$\begin{aligned} \max_{\hat{z}^f} \quad & \sum_{t=0}^{\infty} \hat{p}_t y_t^f - \hat{w}_t n_t^f - \hat{r}_t^k k_t^f \\ \text{s.t.} \quad & y_t^f \leq F(k_t^f, n_t^f) \quad \forall t \geq 0 \\ & y_t^f, n_t^f, k_t^f \geq 0 \quad \forall t \geq 0 \end{aligned}$$

3) Markets clear $\forall t \geq 0$

$$\sum_{i \in I} \hat{h}_t^i = \hat{k}_t^f \quad \forall t \geq 0$$

$$\sum_{i \in I} \hat{n}_t^i = \hat{n}_t^f \quad \forall t \geq 0$$

$$\sum_{i \in I} (\hat{c}_t^i + \hat{x}_t^i) = F(\hat{k}_t^f, \hat{n}_t^f) = \hat{y}_t^f \quad \forall t \geq 0$$

By the strict concavity of U , each consumer's utility maximization problem has a unique maximizer $\hat{z}^{H^i}$. Additionally, since all consumers have identical preferences and endowments, the utility maximization problem is the same for all consumers

(the objective function and constraints are the same).

⚡ Technically, only $\hat{c}_t^i, \hat{x}_t^i, \hat{n}_t^i$ are unique since investing is a zero profit activity (it doesn't matter who makes investment decisions). But in one eqbm: $\hat{k}_t^i = \hat{k}_t^f, \hat{x}_t^i = \hat{x}_t^f \quad \forall t.$

By the uniqueness of $\{\hat{c}_t^i, \hat{l}_t^i, \hat{n}_t^i\}_{t=0}^{\infty} \forall i$, and since all consumers solve the same problem, we get that $\{\hat{c}_t^i, \hat{n}_t^i, \hat{l}_t^i\}_{t=0}^{\infty} = \{\hat{c}_t^j, \hat{n}_t^j, \hat{l}_t^j\}_{t=0}^{\infty} \forall i, j$ in any equilibrium.

We also know that in one of the equilibria, we will have $\hat{x}_t^i = x_t^j$, $\hat{k}_t^i = \hat{k}_t^j \forall i, j, \forall t$.

We will focus on the equilibrium where all agents have identical allocations; $\hat{z}^{H,i} = \hat{z}^{H,j} \forall i, j$.

Let the superscript "A" denote aggregate quantities and superscript "H" denote household/consumer quantities:

Thus, in this equilibrium:

$$\hat{C}_t^A = \sum_{i=1}^N \hat{C}_t^i = \sum_{i=1}^N \hat{C}_t^H = N \hat{C}_t^H \quad (\text{since } \hat{C}_t^i = \hat{C}_t^j = \hat{C}_t^H \forall i, j)$$

$$\Leftrightarrow \hat{C}_t^H = \frac{\hat{C}_t^A}{N} \quad \forall t \geq 0$$

$$\hat{n}_t^A = \sum_{i=1}^N \hat{n}_t^i = \sum_{i=1}^N \hat{n}_t^H = N \hat{n}_t^H \quad (\text{since } \hat{n}_t^i = \hat{n}_t^j = \hat{n}_t^H \forall i, j)$$

$$\Leftrightarrow \hat{n}_t^H = \frac{\hat{n}_t^A}{N} \quad \forall t \geq 0$$

⋮

And the same holds for all other quantities.

Clearly, under these assumptions, there is no inequality across any of the variables in this equilibrium; all consumers have the same amount of consumption, leisure, work, investment, and capital stock. All consumers have a fixed share ($1/N$) of the aggregate quantities in equilibrium.

Because of the properties of this equilibrium, we can solve for the equilibrium household quantities $\{\hat{c}_t^H, \hat{n}_t^H, \hat{l}_t^H, \hat{x}_t^H, \hat{k}_t^H\}_{t=0}^{\infty}$ by calculating an ADE

for only one representative consumer.

The ADE for the representative consumer economy will face the same maximization problem as in the N person economy, but the firm problem & market clearing will be different.

The firm will choose $\{\hat{y}_t^F/N, \hat{k}_t^F/N, \hat{n}_t^F/N\}_{t=0}^{\infty} \equiv \frac{1}{N} \hat{z}^F$

where $\hat{z}^F$ is from N consumer economy. Market clearing will be:

$$\hat{n}_t^H = \frac{\hat{n}_t^F}{N} \quad \forall t \geq 0; \quad \hat{k}_t^H = \frac{\hat{k}_t^F}{N} \quad \forall t \geq 0; \quad \hat{c}_t^H + \hat{x}_t^H = F(\hat{k}_t^F/N, \hat{n}_t^F/N) \quad \forall t \geq 0$$

Proposition: Given the aforementioned assumptions,
 $\{\{\hat{z}^i\}_{i \in I}, \hat{z}^f, \hat{P}\}$ is an ADE with N consumers
 if and only if $\{\hat{z}^H, \frac{1}{N} \hat{z}^f, \hat{P}\}$ is an ADE with
 one representative consumer (given $k_0, \bar{n}_t^f$).

↳ Proof sketch: First, check that market clearing in one
 equilibrium implies market clearing in the other:

$$\sum_{i=1}^N \hat{n}_t^i = N \hat{n}_t^H = \hat{n}_t^f \Leftrightarrow \hat{n}_t^H = \frac{\hat{n}_t^f}{N} \quad \checkmark$$

$$\sum_{i=1}^N \hat{k}_t^i = N \hat{k}_t^H = \hat{k}_t^f \Leftrightarrow \hat{k}_t^H = \frac{\hat{k}_t^f}{N} \quad \checkmark$$

$$\sum_{i=1}^N (\hat{c}_t^i + \hat{x}_t^i) = N(\hat{c}_t^H + \hat{x}_t^H) = F(\hat{k}_t^f, \hat{n}_t^f)$$

$$\Leftrightarrow \hat{c}_t^H + \hat{x}_t^H = \frac{1}{N} F(\hat{k}_t^f, \hat{n}_t^f) \stackrel{\text{CRS}}{=} F\left(\frac{\hat{k}_t^f}{N}, \frac{\hat{n}_t^f}{N}\right) \quad \checkmark$$

Now, check the firm prices:

$$\begin{aligned} \hat{r}_t^k &= \hat{p}_t F_k(\hat{k}_t^f, \hat{n}_t^f) = \hat{p}_t F_k\left(\frac{\hat{k}_t^f}{N}, \frac{\hat{n}_t^f}{N}\right) \\ \hat{w}_t &= \hat{p}_t F_n(\hat{k}_t^f, \hat{n}_t^f) = \hat{p}_t F_n\left(\frac{\hat{k}_t^f}{N}, \frac{\hat{n}_t^f}{N}\right) \end{aligned} \quad \left. \begin{array}{l} \text{Since } F \text{ is} \\ \text{homogeneous of} \\ \text{degree 1, then} \\ \text{partial derivative} \\ \text{is homogeneous} \\ \text{of degree 0.} \end{array} \right\}$$

Utility maximization problem across
 the two equilibria will also be the
 same, since the objective functions +
 budget constraints (same prices + endowments) are identical.
 So all agents optimally choose $\hat{z}^H$ across both equilibria.

We have completed the proof sketch. $\square$

So to solve for household quantities in this setup, we only need to calculate an ADE for a representative consumer, rather than an ADE for N consumers. Given our assumptions, we also know that the allocation from the representative consumer is Pareto efficient. Given the concavity of U and convexity of the production set, we can use the following SPP to calculate the representative consumer's equilibrium allocation $\{\hat{c}_t^H, \hat{n}_t^H, \hat{l}_t^H, \hat{k}_t^H, \hat{x}_t^H\}_{t=0}^{\infty}$:

$$\begin{aligned}
 & \max_{\{c_t^H, n_t^H, l_t^H, x_t^H, k_t^H\}} \sum_{t=0}^{\infty} \beta^t u(c_t^H, l_t^H) \\
 & \text{s.t.} \quad c_t^H + x_t^H \leq F(k_t^H, n_t^H) \quad \forall t \geq 0 \\
 & \quad k_{t+1}^H \leq x_t^H + (1-\delta)k_t^H \quad \forall t \geq 0 \\
 & \quad l_t^H + n_t^H \leq \bar{n}_t^H \quad \forall t \geq 0 \quad \left(\bar{n}_t^H = \bar{n}_t^j = \bar{n}_t^H \quad \forall i, j \right) \\
 & \quad \bar{k}_0^H \text{ given} \quad \left(k_0^i = k_0^j = k_0^H \quad \forall i, j \right) \\
 & \quad k_t^H, c_t^H, n_t^H, l_t^H \geq 0 \quad \forall t \geq 0
 \end{aligned}$$

Method 2: Homothetic Aggregation

Required Assumptions

- Preferences: There are $N \in [1, \infty)$ consumers.

$$\text{Let } U^i(\{c_t^i, l_t^i\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta_i^t u_i(c_t^i, l_t^i)$$

$$\text{such that } \beta_i = \beta_j = \beta \quad \forall i, j$$

(i.e. all consumers' discount factors are same)

and $u_i = u_j = u \quad \forall i, j$ (i.e. all consumers' period utility is the same).

Thus $U^i = U^j = U \quad \forall i, j$, so that all consumers' preferences are identical.

We must also assume that U is homothetic. We will also assume continuity, strict concavity, and strictly increasing.

In this case, U being homothetic is defined as:

$$U(\{\tilde{c}_t, \tilde{l}_t\}_{t=0}^{\infty}) = U(\{\bar{c}_t, \bar{l}_t\}_{t=0}^{\infty})$$

$$\Leftrightarrow U(\{\lambda \tilde{c}_t, \lambda \tilde{l}_t\}_{t=0}^{\infty}) = U(\{\lambda \bar{c}_t, \lambda \bar{l}_t\}_{t=0}^{\infty}) \quad \forall \lambda > 0$$

- Endowments: Initial capital stock k_i and time endowments $\bar{n}_t^i$ can differ across consumers.

Using homothetic aggregation, we can add some heterogeneity to our model, which enables the possibility of observing inequality in the model.

- Firm: Use same assumptions on the firm(s) as in the identical consumer case. Thus there is a representative firm.

This completes our assumptions required for homothetic aggregation.

The idea for this method is that since all consumers are identical and have homothetic preferences, we can calculate an ADE with a representative consumer

and given aggregate endowments. Because of the homothetic

preferences, we can then calculate for individual equilibrium allocations as a share of the aggregate allocation; and the share is determined by each consumer's share of total wealth.

A sufficient condition for U being homothetic is U being homogeneous of degree n .

U is homogeneous of degree n when

$$U(\{\lambda c_t, \lambda l_t\}_{t=0}^{\infty}) = \lambda^n U(\{c_t, l_t\}_{t=0}^{\infty}) \quad \forall \lambda \geq 0.$$

It is straightforward to see that implies homotheticity (try it!).

To see why we can use homothetic aggregation, we will go through a simple example.

Suppose we have a two good economy (x and y) and that consumer i has wealth of 1 ($w_i = 1$):

$$\begin{aligned} \max_{x_i, y_i} \quad & U_i(x_i, y_i) \\ \text{s.t.} \quad & p_x x_i + p_y y_i \leq 1 = w_i \end{aligned}$$

The solution to this problem is the demand function:

$$\left(x_i(\underbrace{1}_{w_i}), y_i(\underbrace{1}_{w_i}) \right) \quad \text{This is just a two dimensional vector-valued function}$$

What happens if we scale wealth by $\lambda \geq 0$?

I claim that if we multiply wealth by $\lambda \geq 0$, then

$$\text{the we get: } (x_i(\lambda \cdot 1), y_i(\lambda \cdot 1)) = (\lambda x_i(1), \lambda y_i(1)).$$

demand function at wealth λ

Claim: $(x_i(\lambda), y_i(\lambda)) = (\lambda x_i(1), \lambda y_i(1)) \quad \forall \lambda \geq 0.$

↳ Proof: Suppose not. $\exists (\tilde{x}, \tilde{y})$ that is affordable

and such that $U(\tilde{x}, \tilde{y}) > U(\lambda x_i(1), \lambda y_i(1)).$

Since $(\tilde{x}, \tilde{y})$ affordable at wealth λ , then

$$p_x \tilde{x} + p_y \tilde{y} \leq \lambda \Leftrightarrow p_x \left(\frac{\tilde{x}}{\lambda}\right) + p_y \left(\frac{\tilde{y}}{\lambda}\right) \leq 1.$$

Thus $\left(\frac{\tilde{x}}{\lambda}, \frac{\tilde{y}}{\lambda}\right)$ is affordable at wealth of 1,

and by homotheticity: $\left. \begin{array}{l} \text{I think we need at a minimum for} \\ U \text{ to be continuous to perform this step.} \end{array} \right\}$

$$U(\tilde{x}, \tilde{y}) > U(\lambda x_i(1), \lambda y_i(1))$$

$$\Leftrightarrow U\left(\frac{\tilde{x}}{\lambda}, \frac{\tilde{y}}{\lambda}\right) > U(x_i(1), y_i(1)).$$

But this contradicts that $(x_i(1), y_i(1))$ is

optimal given wealth of 1, since $(\tilde{x}/\lambda, \tilde{y}/\lambda)$

is affordable at 1 and is strictly preferred.

$$\text{Thus } (x_i(\lambda), y_i(\lambda)) = (\lambda x_i(1), \lambda y_i(1)) \quad \forall \lambda \geq 0. \quad \blacksquare$$

This tells us that at any level of wealth $W_i \geq 0$,

the optimal choice is $(x_i(W_i), y_i(W_i)) = (W_i x_i(1), W_i y_i(1))$

$$= W_i (x_i(1), y_i(1))$$

What if there were N consumers with identical homothetic preferences (so that $x_i = x_j = x$ and $y_i = y_j = y \forall i, j$)?

Let the aggregate demand function be (fixing prices):

$$(x(W_1, W_2, \dots, W_N), y(W_1, W_2, \dots, W_N))$$

$$= \sum_{i=1}^N (x_i(W_i), y_i(W_i))$$

$$= \sum_{i=1}^N (W_i x_i(1), W_i y_i(1))$$

$$= \sum_{i=1}^N W_i (x_i(1), y_i(1))$$

$$= (W_1 + W_2 + \dots + W_N) (x_i(1), y_i(1))$$

$$= \left(\sum_{i=1}^N W_i \right) (x_i(1), y_i(1))$$

$$= (x_i(\sum W_i), y_i(\sum W_i)) = (x(\sum_{i=1}^N W_i), y(\sum_{i=1}^N W_i))$$

Thus the only thing that matters for aggregate demand is total wealth in the economy; the distribution of wealth does not matter.

Thus aggregate demand in an economy with N consumers with wealth W_i is the same as aggregate demand in an economy with one consumer j owning all of the wealth $\bar{W}$ ($= \sum_{i=1}^N W_i$ from previous exercise)

Aggregate demand in this economy is :

$$(x(\bar{W}), y(\bar{W}))$$

$$= (x_j(\bar{W}), y_j(\bar{W}))$$

$$= (\bar{W} x_j(1), \bar{W} y_j(1))$$

$$= \bar{W} (x_j(1), y_j(1)) = \bar{W} (x(1), y(1))$$

Since $\bar{W} = \sum_{i=1}^N W_i$, where W_i came from previous example,

we see that aggregate demand in this economy with only one consumer (with all wealth) is the same as in an economy with N consumers and any distributions of wealth.

So with homothetic preferences, the distribution of wealth does not affect aggregate demand. The total amount of wealth is all that matters. Thus, one "representative" consumer with all wealth makes same choice as N consumers in the aggregate.

Thus, the aggregate demand in an N consumer economy is:

$$(x(W), y(W)) \quad \left(\text{where } W = \sum_{i=1}^N W_i \right)$$

Suppose we know the aggregate demand function, but want to solve for consumer i 's allocation. We can do some algebra using aggregate demand:

$$(x(W), y(W)) = (W) (x_i(1), y_i(1))$$

$$\Leftrightarrow \frac{1}{W} (x(W), y(W)) = (x_i(1), y_i(1))$$

Multiply both sides by W_i :

$$\frac{W_i}{W} (x(W), y(W)) = W_i (x_i(1), y_i(1))$$

consumer i 's demand of x & y given W_i

$$\Leftrightarrow \left(\frac{W_i}{\sum_{i=1}^N W_i} x(W), \frac{W_i}{\sum_{i=1}^N W_i} y(W) \right) = (x_i(W_i), y_i(W_i))$$

Thus we see that consumer i optimally consumes a fixed share of the aggregate demand of good x and good y . This share is

equal to $\frac{W_i}{\sum_{i=1}^N W_i}$; the fraction total wealth they own.

We can apply this logic to our models as long as we assume all consumers have identical homothetic preferences. First, what is "wealth" in ADE?

Consider the budget constraint of consumer i (assume binds):

$$\sum_{t=0}^{\infty} \hat{p}_t (c_t^i + k_{t+1}^i - (1-\delta)k_t^i) = \sum_{t=0}^{\infty} \hat{w}_t n_t^i + \hat{r}_t^k k_t^i$$

Using $l_t^i + n_t^i = \bar{n}_t^i$ and rearranging, we get:

$$\sum_{t=0}^{\infty} \hat{p}_t c_t^i + \hat{w}_t l_t^i = \sum_{t=0}^{\infty} \hat{w}_t n_t^i + (\hat{r}_t^k + \hat{p}_t(1-\delta))k_t^i - \hat{p}_t k_{t+1}^i$$

Since $\hat{p}_t = \hat{r}_{t+1}^k + \hat{p}_{t+1}(1-\delta)$ in equilibrium, we can

show that the RHS is equal to:

$$\text{RHS} = k_0^i \underbrace{(\hat{r}_0^k + (1-\delta)\hat{p}_0)}_{\hat{q}_0} + \sum_{t=0}^{\infty} \hat{w}_t \bar{n}_t^i$$

Thus consumer i 's wealth is $W_i = k_0^i \hat{q}_0 + \sum_{t=0}^{\infty} \hat{w}_t \bar{n}_t^i$

and total wealth ^{in the economy} is $W = \sum_{i=1}^N W_i = \hat{q}_0 \left(\sum_{i=1}^N k_0^i \right) + \sum_{t=0}^{\infty} \hat{w}_t \left(\sum_{i=1}^N \bar{n}_t^i \right)$

So consumer i 's proportion of total wealth in the economy is $\frac{W_i}{W}$.

We see that wealth depends on the present discounted value of the endowments.

Now we need to calculate aggregate demand. As we showed earlier, aggregate demand does not depend on the distribution of wealth because of homothetic preferences.

Thus, if we assumed that there was only one consumer who owned all of the wealth in the N consumer economy (so that their initial capital stock is $\sum_{i=1}^N k_0^i$ and their time endowment $\bar{t}$ is $\sum_{i=1}^N \bar{n}_i^t$), then this

"representative" consumer's optimal quantities would be equivalent to aggregate demand in the N consumer economy.

Given these optimal quantities, the firm's problem and market clearing are identical in the one consumer (with all wealth) economy and the N consumer economy. So we get some sort of equivalence between the one consumer equilibrium and the N consumer equilibrium.

We have the following result:

Proposition: Assume $V^i = U \quad \forall i$ and U is homothetic and that F is CRS.

Let $\left[\{ \hat{p}_t, \hat{r}_t^k, \hat{w}_t \}_{t=0}^{\infty}, \{ \{ \hat{c}_t^i, \hat{x}_t^i, \hat{k}_t^i, \hat{n}_t^i, \hat{l}_t^i \}_{t=0}^{\infty} \}_{i=1}^N, \{ \hat{y}_t^f, \hat{k}_t^f, \hat{n}_t^f \}_{t=0}^{\infty} \right]$

be an ADE given $k_0^i \quad \forall i$ and $\bar{n}_t^i \quad \forall i, \forall t$

Then:

$\left[\{ \hat{p}_t, \hat{r}_t^k, \hat{w}_t \}_{t=0}^{\infty}, \{ \sum_{i=1}^N \hat{c}_t^i, \sum_{i=1}^N \hat{x}_t^i, \sum_{i=1}^N \hat{k}_t^i, \sum_{i=1}^N \hat{n}_t^i, \sum_{i=1}^N \hat{l}_t^i \}_{t=0}^{\infty}, \{ \hat{y}_t^f, \hat{k}_t^f, \hat{n}_t^f \}_{t=0}^{\infty} \right]$

is an ADE given $\sum_{i=1}^N k_0^i$ and $\sum_{i=1}^N \bar{n}_t^i \quad \forall t$. ■

Let $n_i = \frac{W_i}{W}$ be consumer i 's proportion of aggregate wealth.

We have another proposition: ④ - There is a subtlety here; the proposition as stated is not completely true. See last page of this document.

Proposition: Assume $V^i = U \quad \forall i$ and U is homothetic and that F is CRS.

Let $\left[\{ \hat{p}_t, \hat{r}_t^k, \hat{w}_t \}_{t=0}^{\infty}, \{ \hat{c}_t, \hat{x}_t, \hat{k}_t, \hat{n}_t, \hat{l}_t \}_{t=0}^{\infty}, \{ \hat{y}_t^f, \hat{k}_t^f, \hat{n}_t^f \}_{t=0}^{\infty} \right]$

be an ADE with a single homothetic representative consumer

given $\sum_{i=1}^N k_0^i$ and $\sum_{i=1}^N \bar{n}_t^i \quad \forall t$. Then:

$\left[\{ \hat{p}_t, \hat{r}_t^k, \hat{w}_t \}_{t=0}^{\infty}, \{ \{ n_i \hat{c}_t, n_i \hat{x}_t, n_i \hat{k}_t, n_i \hat{n}_t, n_i \hat{l}_t \}_{t=0}^{\infty} \}_{i=1}^N, \{ \hat{y}_t^f, \hat{k}_t^f, \hat{n}_t^f \}_{t=0}^{\infty} \right]$

as an ADE with N consumers given $k_0^i \quad \forall i$ and $\bar{n}_t^i \quad \forall i, \forall t$.

And where $n_i = \frac{W_i}{W}$, with W_i and W defined above. ■

Since the FWT holds in this setup, we know the equilibrium allocation with the representative consumer (given the aggregate endowment) is Pareto Efficient.

This points us to using a social planner's problem.

Given our assumptions on U & F , we know that this equilibrium allocation solves the following planner's problem:

$$\max_{\{c_t, x_t, n_t, l_t, k_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

$$\text{s.t.} \quad c_t + x_t \leq F(k_t, n_t) \quad \forall t \geq 0$$

$$k_{t+1} \leq (1-\delta)k_t + x_t \quad \forall t \geq 0$$

$$l_t + n_t \leq \sum_{i=1}^N \bar{n}_i^i \quad \forall t \geq 0$$

$$\sum_{i=1}^N k_0^i \text{ given}$$

$$c_t, n_t, l_t, k_t \geq 0 \quad \forall t \geq 0$$

Once we solve for the optimal allocation from this

planner's problem, $\{\hat{c}_t, \hat{n}_t, \hat{l}_t, \hat{k}_t, \hat{x}_t\}_{t=0}^{\infty}$, we can

calculate $(w_i/w) = n_i$ and then solve for each consumer's

individual equilibrium quantities:

$$\begin{aligned} \hat{c}_t^i &= n_i \hat{c}_t \\ \hat{l}_t^i &= n_i \hat{l}_t \\ \hat{n}_t^i &= n_i \hat{n}_t \\ \hat{k}_{t+1}^i &= n_i \hat{k}_{t+1} \\ \hat{x}_t^i &= n_i \hat{x}_t \end{aligned}$$

Note that in this model, inequality in initial wealth leads to inequality in what the consumers care about: consumption, leisure, and labor time.

Notice that inequality in these variables is stable over time; all consumers consume a fixed share of the aggregate variables for all of time.

So although there can be inequality in this model (which is great - there obviously exists inequality in real life), there is no inequality dynamics.

We would eventually like to model how the distribution of wealth changes over time, but that is a task for later courses.

What if can't aggregate?

As long as the CE allocation is Pareto efficient (i.e. the FWT holds) and if U and F satisfy certain conditions (what are they?), we know that the equilibrium allocation will solve a planner's problem for some set of Pareto weights ($\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$).

With a single firm and N consumers, the SPP is:

$$\max_{\{\{c_i, n_i, l_i\}_{i \in I}, x_t, k_t^f, n_t^f\}_{t=0}^{\infty}} \sum_{i=1}^N \alpha_i \left(\sum_{t=0}^{\infty} \beta^t u_i(c_t^i, l_t^i) \right)$$

$$\text{s.t.} \quad \left(\sum_{i=1}^N c_t^i \right) + x_t \leq F(k_t^f, n_t^f) \quad \forall t \geq 0$$

$$k_{t+1}^f \leq x_t + (1-\delta)k_t^f \quad \forall t \geq 0$$

$$l_t^i + n_t^i \leq \bar{n}^i \quad \forall i \in I, \forall t \geq 0$$

$$n_t^f \leq \sum_{i=1}^N n_t^i \quad \forall t \geq 0$$

$$k_0^f = \sum_{i=1}^N k_0^i \quad \text{given}$$

$$\text{non-negativity} \quad \forall t \geq 0$$

We can then use the Negishi method to solve for the Pareto weights such that transfers are zero, and then use those Pareto weights to solve for the equilibrium allocation. The prices will come from the Lagrange multipliers from the planner's problem.

If you want to review the Negishi method, please refer to Professor Dirk Krueger's notes.

⚡ As noted at the top of page 52 of Professor Jones' notes, there is a subtlety here. Do not feel like you are required to read this section, as this subtlety will never come up in problem sets or exams.

In this ADE, we need to allow the possibility of $l_t^i > \bar{n}_t^i$ or $n_t^i > \bar{n}_t^i$. Thus, we could drop the constraint $l_t^i + n_t^i \leq \bar{n}_t^i$ from each utility max. problem and add an aggregate time constraint to the market clearing conditions ($\sum_{i=1}^N l_t^i + n_t^i \leq \sum_{i=1}^N \bar{n}_t^i \forall t$). However, this is beyond the scope of the class, so you will not be asked about this subtlety on exams. Any questions about homothetic preferences will avoid this issue (like 2021 midterm Q1(f)).