

8106 Recitation 1

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1 Overview

In these notes, we review what we learned in 8105 about dynamic programming.¹ We also preview the consumer aggregation results we will discuss later in the course.

2 Dynamic Programming Review

In 8105, we primarily focused on the infinitely lived agent model with a single consumer type. We will focus on this specific example here as well. We will use the same environment for the single-sector growth model that we have used many times before:

- Single good in each period that can be consumed or saved as capital
- The consumer has the following endowments:
 - One unit of time per period
 - k_0 units of initial capital
- The consumer's utility is given by our standard discounted CRRA utility function ($\sigma \in (0, 1)$) with discount factor $\beta \in (0, 1)$:

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \text{ with } u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$$

- Cobb-Douglas production function with $\alpha \in (0, 1)$:

$$F(k_t, l_t) = \theta k_t^\alpha l_t^{1-\alpha}$$

- Standard law of motion of capital with depreciation $\delta \in (0, 1]$:

$$k_{t+1} \leq x_t + (1 - \delta)k_t$$

- Thus the feasibility constraint is:

$$c_t + k_{t+1} - (1 - \delta)k_t \leq \theta k_t^\alpha l_t^{1-\alpha}$$

¹These notes are primarily adapted from Amol Amol's teaching notes, along with Professor Larry Jones', Conor Ryan's, Alex Wurdinger's notes. Theorems and definitions come directly from Stokey, Lucas, and Prescott's *Recursive Methods in Economic Dynamics* (which I call SLP). I do not claim credit for these notes and errors are my own.

Since leisure is not in the utility function, we know that $\hat{l}_t = 1$ for all periods. We can write the social planner's problem as:

$$\begin{aligned} \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t.} \quad & c_t + k_{t+1} - (1 - \delta)k_t \leq \theta k_t^\alpha \quad \forall t \\ & k_{t+1}, c_t \geq 0 \quad \forall t \\ & k_0 \in K \subseteq \mathbb{R}_+ \text{ given} \end{aligned}$$

The Bellman equation for this problem is:

$$\begin{aligned} v(k) &= \max_{c, k'} [u(c) + \beta v(k')] \\ \text{s.t.} \quad & c + k' \leq \theta k^\alpha \\ & c, k' \geq 0 \end{aligned}$$

We can also write the social planner's problem and the Bellman equation in SLP's preferred format. The **sequence problem (SP)** form of the above problem is:²

$$\begin{aligned} \max_{\{k_{t+1}\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t F(k_t, k_{t+1}) \\ \text{s.t.} \quad & k_{t+1} \in \Gamma(k_t) \quad \forall t \\ & k_0 \in K \text{ given} \end{aligned}$$

Where the period return function ($F : K \times K \rightarrow \mathbb{R}$) is defined as:³

$$\begin{aligned} F(k_t, k_{t+1}) &= \max_{c_t} u(c_t) \\ \text{s.t.} \quad & c_t + k_{t+1} - (1 - \delta)k_t \leq \theta k_t^\alpha \\ & c_t \geq 0 \end{aligned}$$

Given that $u(c_t)$ is strictly increasing in c_t , we know that the feasibility constraint will bind. Thus we immediately see that:

$$F(k_t, k_{t+1}) = u(\theta k_t^\alpha + (1 - \delta)k_t - k_{t+1})$$

And the feasibility correspondence is defined as:

$$\Gamma(k_t) = \{k_{t+1} \in K \mid 0 \leq k_{t+1} \leq \theta k_t^\alpha + (1 - \delta)k_t\}$$

The **functional equation (FE)** for this problem:

$$v(k) = \max_{k' \in \Gamma(k)} [F(k, k') + \beta v(k')], \quad \forall k \in K$$

²Note that we do not see c_t anywhere in the SP form. This is because it is not a dynamic variable. In other words, it is not today's state variable (like k_t), nor is it tomorrow's state variable (like k_{t+1}). It is just a "control" variable that only effects utility in the current period.

³In general, we first have to ensure that the solution to this maximization problem actually exists.

Where the period return function and feasibility correspondence are the same as above (sans time subscripts):

$$F(k, k') = u(\theta k^\alpha + (1 - \delta)k - k')$$

$$\Gamma(k) = \{k' \in K \mid 0 \leq k' \leq \theta k^\alpha + (1 - \delta)k\}$$

In most of these problems, our ultimate goal is to calculate an ADE or SME (or some other equilibrium concept). If our utility function and production function meet certain assumptions, we can apply the welfare theorems and thus use the solution to the social planner's problem to calculate the equilibrium allocations. But the social planner's problem is very difficult to solve in the sequence form. Under the conditions outlined in SLP chapter 4.1, the FE has the same solution as the SP. Why do we care? Because the FE is generally much, much easier to solve than the SP.

Next, I outline some basic steps on how to approach these types of problems.

1. Using the Bellman equation above, define the operator $T : B(K) \rightarrow B(K)$:

$$(Tw)(k) = \max_{k' \in \Gamma(k)} [u(\theta k^\alpha + (1 - \delta)k - k') + \beta w(k)]$$

where $w : K \rightarrow \mathbb{R}$ is a function in $B(K)$, the space of bounded functions (and thus w is bounded).

2. Use Blackwell's sufficient conditions to show that the operator $T : B(K) \rightarrow B(K)$ is a contraction mapping with modulus β .

Definition (Contraction Mapping): Let (S, ρ) be a metric space and $T : S \rightarrow S$ be a function mapping S into itself. T is a contraction mapping (with modulus β) if for some $\beta \in (0, 1)$,
 $\rho(Tv, Tw) \leq \beta \rho(v, w)$, for all $v, w \in S$.

Theorem (Blackwell's sufficient conditions): Let $K \subseteq \mathbb{R}$, and let $B(K)$ be a space of bounded functions $v : K \rightarrow \mathbb{R}$, with the sup norm. Let $T : B(K) \rightarrow B(K)$ be an operator satisfying:

- a. **Monotonicity:** If $f, g \in B(K)$ and $f(k) \leq g(k) \forall k \in K$, then
 $(Tf)(k) \leq (Tg)(k) \forall k \in K$.

- b. **Discounting:** There exists some $\beta \in (0, 1)$ such that:

$$[T(f + a)](k) \leq (Tf)(k) + \beta a, \quad \forall f \in B(K), \quad a \geq 0, \quad k \in K.$$

$$(\text{Where } (f + a)(x) = f(x) + a.)$$

Then T is a contraction with modulus β . ■

In our case, it is straightforward to prove that the operator defined above satisfies Blackwell's sufficient conditions. Refer to my previous notes to see the proof (I might add it here later).

3. Since the operator is a contraction mapping and $(B(K), \rho)$ is a complete metric space (where ρ is the sup norm), we can apply the contraction mapping theorem (CMT).

Definition (Complete Metric Space): A metric space (S, ρ) is complete if every Cauchy sequence in S converges to an element in S .

Definition (Convergence): A sequence $\{x_n\}_{n=0}^{\infty}$ in S converges to $x \in S$, if for each $\epsilon > 0$, there exists N_ϵ such that
 $\rho(x_n, x) < \epsilon, \quad \forall n \geq N_\epsilon$

Definition (Cauchy Sequence): A sequence $\{x_n\}_{n=0}^{\infty}$ in S is a Cauchy sequence (satisfies the Cauchy criterion) if for each $\epsilon > 0$, there exists N_ϵ such that
 $\rho(x_n, x_m) < \epsilon, \quad \forall n, m \geq N_\epsilon$.

The CMT tells us that the contraction T has a unique fixed point and that iterating on an initial guess will bring us closer and closer to that fixed point.

Theorem (Contraction Mapping Theorem): If (S, ρ) is a complete metric space and $T : S \rightarrow S$ is a contraction mapping with modulus β , then:

- a. T has exactly one (unique) fixed point v in S , and
 - b. for any $v_0 \in S$, $\rho(T^n v_0, v) \leq \beta^n \rho(v_0, v)$, $n = 0, 1, 2, \dots$ ■
4. Using the CMT and our operator, we can start with an initial guess for the functional equation, a function $v_0 : K \rightarrow \mathbb{R}$ where $v_0 \in B(K)$. Plugging this into the operator gives us $v_1 = T(v_0)$. Using this “method of successive approximations,”⁴ we can keep iterating $v_{n+1} = T(v_n)$. With each additional iteration v_{n+1} gets closer to the fixed point $v = T(v)$.

Ideally, we want $n \rightarrow \infty$ to get $\lim_{n \rightarrow \infty} v_n = v$ exactly. However, when we perform value function iteration on the computer we usually pick a tolerance ϵ and stop iterating once $\max_{k \in K} |v_{n+1}(k) - v_n(k)| < \epsilon$. I usually choose $\epsilon = 10^{-8}$. Exercise 3.9 in SLP gives us an upper bound for how far away $v_n = T^n v_0$ is from v when we reach the tolerance ϵ at iteration $n + 1$:

$$\max_{k \in K} |v_n(k) - v(k)| \leq \frac{1}{1 - \beta} \max_{k \in K} |v_n(k) - v_{n+1}(k)|$$

Suppose we pick our tolerance of $\epsilon = 10^{-8}$ and the discount factor in the problem is $\beta = 0.95$. If the tolerance is met at iteration $N + 1$, the inequality tells us that the distance between v_N and v is at most $\epsilon / (1 - \beta) = 10^{-8} / 0.05 = 20 \cdot 10^{-8}$. That is:

$$\max_{k \in K} |v_n(k) - v(k)| < \frac{\epsilon}{1 - \beta} = 20 \cdot 10^{-8}$$

5. Once we obtain the fixed point $v : K \rightarrow \mathbb{R}$, either through value function iteration or guess and verify, we can characterize the solution $g_k(k) = \hat{k}'$ (policy function) using the first order condition and the envelope condition.

First order condition: $u'(\theta k^\alpha + (1 - \delta)k - \hat{k}') = \beta v'(\hat{k}')$

- The first order condition “equates the marginal utility of consuming current output to the marginal utility of allocating it to capital and enjoying augmented consumption next period.” (SLP, pg. 14)

Envelope condition: $v'(k) = (\alpha \theta k^{\alpha-1} + (1 - \delta))u'(\theta k^\alpha + (1 - \delta)k - \hat{k}')$

⁴See SLP pages 39-40.

- The envelope condition “states that the marginal value of current capital, in terms of total discounted utility, is given by the marginal utility of using the capital in current production and allocating its return to current consumption.” (SLP, pg. 14)

Combining these will give us our usual Euler equation:

$$u'(\theta k^\alpha + (1 - \delta)k - \hat{k}') = \beta v'(\hat{k}') = \beta(\alpha \theta \hat{k}'^{\alpha-1} + (1 - \delta))u'(\theta \hat{k}'^\alpha + (1 - \delta)\hat{k}' - \hat{k}'')$$

6. We can verify that the solution to the FE and SP are equivalent using chapter 4.1 of SLP. We call this idea the “Principle of Optimality.” The below properties of our current problem are important for satisfying the Principle of Optimality:

- Utility function: Continuous and bounded below
- Production function: Continuous, strictly increasing, concave, satisfies Inada
- Parameters: $\beta \in (0, 1)$; $\delta \in (0, 1]$; $\alpha \in (0, 1)$

We can then ensure $F : K \times K \rightarrow \mathbb{R}$ is bounded above by bounding the set of possible capital values K using the maximum sustainable capital ($\tilde{k}$), and set $K = [0, \max\{\tilde{k}, k_0\}]$

7. Since the Principle of Optimality holds, we can use the policy function from the FE to solve for the solution to the SP $\{\hat{c}_t, \hat{k}_{t+1}\}_{t=0}^\infty$ (given initial capital stock k_0).
8. Since the utility function is strictly increasing, concave, and continuous, and since the production function is increasing, concave, and continuous, we know that both the First Welfare Theorem (FWT) and Second Welfare Theorem (SWT) hold.

By the FWT, we know that an ADE allocation is Pareto Efficient. Thus we know that the ADE allocation solves a social planner’s problem. Let the solution to this solution planner’s problem be $\{\hat{c}_t, \hat{k}_{t+1}\}_{t=0}^\infty$. By the SWT, there exists prices and transfers such that $\left(\{\hat{p}_t, \hat{r}_t^k, \hat{w}_t\}_{t=0}^\infty, \hat{T}_1, \{\hat{y}_t^f, \hat{k}_t^f\}_{t=0}^\infty, \{\hat{c}_t, \hat{k}_{t+1}\}_{t=0}^\infty\right)$ form an ADE with transfers.

9. Since we only have one agent in this setup, we know that there are no transfers, $\hat{T}_1 = 0$. Thus the solution to the social planner’s problem is the equilibrium allocation. As we have seen many times before, we can use the household FOCs and firm’s problem to get the prices:

$$\begin{aligned}\hat{p}_0 &= 1 \\ \frac{u'(\hat{c}_t)}{\beta u'(\hat{c}_{t+1})} &= \frac{\hat{p}_t}{\hat{p}_{t+1}} \\ \hat{r}_t^k &= \hat{p}_t \alpha \theta \hat{k}_t^{\alpha-1} \\ \hat{w}_t &= \hat{p}_t (1 - \alpha) \theta \hat{k}_t^\alpha\end{aligned}$$

And using our equivalence results from 8105, we can also obtain a SME.

In this example, we assumed that there was one type of consumer. However, we can be more general. There are two consumer “aggregation” methods that we discuss in 8106. Both of these methods allow us to solve a planner’s problem for one **representative consumer**, rather than solving the social planner’s problem with multiple consumers. I give a preview below.⁵

If there are multiple types of consumers and they all have the same preferences and endowments (they are all **identical**), we can treat them as a single (“representative”) consumer and solve the social planner’s problem for only one consumer. The idea is that since all agents are identical, they will make identical decisions. Thus allocations are the same for all consumers.

If endowments are different but preferences are identical and **homothetic**, we can also solve a social planner’s problem with a representative consumer. Our constraints for this problem include aggregate feasibility and the aggregate labor/leisure constraint $l_t + n_t \leq \sum_{i=1}^N \ell_t^i$, with the aggregate initial capital stock $k_0 = \sum_{i=1}^N k_0^i$ given. Individual allocations will be the values of the aggregate social planner’s problem allocation weighted by each agent’s share of aggregate wealth.

When preferences are different across consumers, we can’t use either of the representative consumer approaches. However, we can still solve the social planner’s problem by maximizing the weighted (with Pareto weights) sum of lifetime utilities subject to feasibility. In this case, we use the aggregate initial capital stock $k_0 = \sum_{i=1}^N k_0^i$. We can then use the Negishi method to solve for each agent’s equilibrium allocation.⁶ But since we are maximizing the utility of multiple consumers, this problem is much more complicated. We should **always** use the representative consumer approach if we can!

⁵See Section 4 of Professor Jones’ notes. There are some other key conditions that must be met for these aggregation results to hold.

⁶This problem is actually more difficult than just solving the Negishi method, since the planner’s problem only solves for aggregate capital stock. So you also need to determine each consumer’s share of the capital stock for all periods.