

Macroeconomic Theory (Econ 8106)

Larry Jones

Fall 2022

Problem Set 2

Questions 1-5 due on November 15 at 2:30 PM

You can submit Questions 6-8 later (there is no specific deadline as of now).

Question 1: Guess and Verify - Optimal Growth with Leisure

Consider the problem of choosing sequences $\{(c_t, k_{t+1}, l_t, n_t)\}_{t=0}^{\infty}$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t (\log c_t + \gamma \log l_t)$$

subject to:

$$c_t + k_{t+1} \leq \theta k_t^\alpha n_t^{1-\alpha},$$

$$n_t + l_t \leq 1,$$

$$c_t, k_{t+1}, l_t, n_t \geq 0$$

$$k_0 \text{ given}$$

where $0 < \alpha < 1$, $0 < \beta < 1$ and $\theta > 0$.

- (a) Write down the Euler equation, intratemporal marginal rate of substitution between consumption and leisure, and transversality condition for the problem above. Interpret each of these conditions.
- (b) Write down the Bellman equation for this problem. What are the control variables and what is the state variable?
- (c) Using the Bellman equation, characterize the optimal policy function through the first-order and envelope conditions. Derive the Euler equation and intratemporal marginal rate of substitution between consumption and leisure (they should be the same as what you derived in (b)).
- (d) Guess that the value function $V(k)$ has the form $a_0 + a_1 \log k$. Find the analytical solution for this value function $V(k)$ and the policy functions $g_c(k)$, $g_k(k)$, $g_l(k)$, $g_n(k)$. (Hint: the policy function for labor is constant.) Verify that these policy functions satisfy the conditions in part (b).
- (e) **Extra Credit - Can be handed in later:** Choose a set of parameter values (for β ,

θ , α , γ , and k_0) and create a 100 period time series using each of the policy functions above.

- (f) Suppose now that there are equal populations of 2 types of consumers, with the same discount factor. They have different period utility functions— $\log c_t + \gamma_1 \log l_t$ and $\log c_t + \gamma_2 \log l_t$. Does the equilibrium allocation for this new economy solve a dynamic programming problem like that in part (a)? Carefully explain why or why not (make sure you think about the welfare theorems, not only the Principle of Optimality). If it does solve such a problem, write down the Bellman equation.

Question 2: Homothetic-Homogeneous Problem

Consider the following problem:

$$v(k_0) = \max \sum_{t=0}^{\infty} \beta^t \frac{c_t^\gamma}{\gamma}$$

subject to:

$$c_t + x_t \leq Ak_t,$$

$$k_{t+1} \leq (1 - \delta)k_t + x_t,$$

$$c_t, k_{t+1} \geq 0,$$

$$k_0 > 0 \text{ given}$$

- (a) Characterize the homogeneity properties of the optimal decision rules in initial capital stock, k_0 . I.e. prove that if $\{c_t, k_{t+1}, x_t\}_{t=0}^{\infty}$ is the optimal allocation given k_0 , then $\{\eta c_t, \eta k_{t+1}, \eta x_t\}_{t=0}^{\infty}$ is the optimal allocation given ηk_0 (where $\eta > 0$). Clearly explain each step in the proof.
- (b) Show that the value function is homogeneous of degree γ in the initial capital stock.
- (c) Will these results also hold for the functional equation? Why or why not?

Question 3: Steady States

For this problem, you will want to read Professor Jones' notes; around pages 63-71.

$$\max_{\{c_t, l_t, n_t, x_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to:

$$c_t + x_t \leq F(k_t, n_t)$$

$$k_{t+1} \leq (1 - \delta)k_t + x_t$$

$$n_t + l_t \leq 1$$

$$c_t, k_{t+1}, l_t, n_t \geq 0$$

$$k_0 \text{ given}$$

- (a) Write the functional equation and characterize the optimal policy function through the first-order and envelope condition.
- (b) Use the conditions in (a) to show that the optimal policy function is strictly increasing. (Clearly state the assumptions you need on u , F , δ , β .) (**Hint:** You will want SLP Theorems 4.8 and 4.11 to hold.)
- (c) Define a steady-state for this problem. Use the conditions in (a) to show that there is a unique positive steady-state. (Clearly state the assumptions you need on u , F , δ , β .)
- (d) Define a globally asymptotically stable system. What does this mean in regard to the steady state?
- (e) Now, suppose $u(c_t) = c_t^{1-\sigma}/(1-\sigma)$ and $F(k_t, n_t) = \theta k_t^\alpha n_t^{1-\alpha}$. What is the steady state in this case? Does the steady state change if we instead had $u(c_t) = \log(c_t)$?
- (f) For the case in (e), what happens to the steady state value of capital when you change each parameter, holding all else constant? The parameters are $(\beta, \delta, \theta, \alpha)$. What happens to the steady state value of consumption? Do these results make sense? Given that you started in the original steady state, this gives you some insight into what would happen to the capital stock/consumption time series if there was a “shock” to a parameter (given that the agent can re-optimize after the shock).

Question 4: SLP Section 4.2

For this problem, please read chapter 4.2 of Stokey, Lucas, and Prescott’s (SLP) *Recursive Methods in Economic Dynamics*. It is available online through the university library. This is the same model you used last week to go through chapter 4.1 of SLP.

Suppose we have a single infinitely lived agent with preferences:

$$\sum_{t=0}^{\infty} \beta^t c_t^\rho$$

where $\beta \in (0, 1)$ and $\rho \in (0, 1)$. The production function is Cobb-Douglas (with labor inelastically supplied at 1):

$$F(k, 1) = \theta k_t^\alpha$$

where $\theta > 0$ and $\alpha \in (0, 1)$. The law of motion of capital is:

$$k_{t+1} \leq i_t + (1 - \delta)k_t$$

Where $\delta \in (0, 1]$. Therefore, the feasibility constraint is:

$$c_t + k_{t+1} - (1 - \delta)k_t \leq \theta k_t^\alpha$$

As always, consumption and capital stock must be non-negative.

- (a) Write down the functional equation using the model above.

- (b) State Assumption 4.3. What is X in the above environment? What are Γ and F ? Explain why the model above satisfies A4.3.
- (c) State Assumption 4.4. What is A in the model above? Explain why our current model satisfies A4.4.
- (d) Explain why if A4.3-A4.4 are satisfied, then A4.1-A4.2 are satisfied.
- (e) State Theorem 4.6.
 - What does this theorem tell us about the properties of v (the functional equation (1) in section 4.2) and v^* (the supremum function from section 4.1 (pg. 70))?
 - Pairing this with Theorem 4.5, what does this tell us about the existence of an optimal plan? And why? (**Hint:** Think about Weierstrass Theorem).
- (f) State Assumption 4.5. Explain why it is satisfied in the model above.
- (g) State Assumption 4.6. Explain why it is satisfied in the model above.
- (h) State Corollary 1 on page 52 (this is used in the proof of Theorem 4.7).
- (i) State Theorem 4.7. How is Corollary 1 used in the proof of this theorem?
- (j) State Assumption 4.7. Explain why it is satisfied in the model above.
- (k) State Assumption 4.8. Explain why it is satisfied in the model above.
- (l) State theorem 4.8. Explain how Corollary 1 is used in the proof of this theorem. What does it mean for G to be a “single valued function?”
- (m) State Theorem 4.10. Explain why Figure 4.1 illustrates the idea behind this theorem.
- (n) State Assumption 4.9. Explain why it is satisfied in the model above.
- (o) State Theorem 4.11. Why do we care about the differentiability of v (how could it potentially help us learn the properties of the policy function)?
- (p) Create a table to summarize the results of the assumptions and theorems above. This table should show specific properties of v and G and which assumption(s) and corresponding theorems are required for each.

Question 5: Optimal Growth (SLP Exercise 5.1)

Complete Exercise 5.1 parts (a)-(c) from SLP (pages 103-105).

Question 6: Theorem of the maximum

- (a) Read pages 55-57 of SLP. Define a correspondence.
- (b) Define a upper hemi-continuous (uhc) correspondence and lower hemi-continuous (lhc) correspondence. What does it mean for a correspondence to be continuous?
- (c) Look at Figure 3.2 on page 57 of SLP. Explain why:
 - (a) The correspondence is lhc but not uhc at x_1
 - (b) The correspondence is uhc but not lhc at x_2
 - (c) The correspondence is continuous everywhere else
- (d) Consider the feasibility correspondence:

$$\Gamma(k) = \{k' \in \mathbb{R} | 0 \leq k' \leq \theta k^\alpha\}$$

Prove that $\Gamma(\cdot)$ is continuous.

- (e) State the Theorem of the Maximum. (You can refer to SLP Ch. 3.3)
Suppose $f(\cdot)$ is a continuous real valued function. What can be said about $g : \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$g(k) = \arg \max_x f(x) \\ \text{s.t. } x \in \Gamma(k).$$

- (f) Suppose now that $f(\cdot)$ is strictly quasi-concave. What can be said about g ?

Question 7: Blackwell's Sufficient Conditions and The Contraction Mapping Theorem

Refer to chapter 3 of SLP. When you are asked to prove something, explain each step in your own words. The math will be the same as in the book, but you need to explain what the math means.

Let $X \subset \mathbb{R}^l$ and $B(X)$ be the set of bounded functions with the *sup* norm.

- (a) State Blackwell's Sufficient Conditions for an operator $T : B(X) \rightarrow B(X)$.
- (b) Define a contraction. Prove that if T satisfies Blackwell's conditions, then T is a contraction.
- (c) Define a fixed point for T . State and prove the Contraction Mapping Theorem.

Question 8: The One Sector Growth Model

Consider the following single-sector growth problem.

$$\begin{aligned} & \max_{\{c_t, l_t, n_t, x_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t, l_t) \\ & \text{subject to:} \\ & c_t + x_t \leq f(k_t, n_t), \\ & k_{t+1} \leq (1 - \delta)k_t + x_t, \\ & n_t + l_t \leq 1, \\ & c_t, k_{t+1}, l_t, n_t \geq 0 \\ & k_0 \text{ given} \end{aligned}$$

At each step, try to provide the “minimal” assumptions required.

- (a) Write down the conditions on U and f such that this social planner’s problem can be written as a dynamic programming problem (i.e. the Principle of Optimality is satisfied). Write down the functional equation.
- (b) Write down the conditions on f and δ such that there exists a maximal sustainable level of capital $\bar{K}$, which will be when the consumer consumes nothing and works as much as possible. What equation does $\bar{K}$ satisfy?
- (c) Write down the conditions on U, f, β and δ such that Assumption 4.3 and Assumption 4.4 (SLP, pp. 78) hold. Show these conditions satisfy A4.3 and A4.4.
- (d) Let v and G be as defined in SLP section 4.2. Prove that, under these conditions, v is a bounded and continuous function. If you use any theorems from chapter 4 of SLP, you must prove them in your own words.
- (e) Write down the extra conditions on U and f such that Assumption 4.5 and Assumption 4.6 (SLP, pp. 80) hold. Show that these conditions imply the assumptions. What can you say about v now? What about G ?
- (f) Write down the extra conditions on U and f such that Assumption 4.7 and Assumption 4.8 (SLP, pp. 80) hold. Show that these conditions imply the assumptions. What can you say about v and G now?
- (g) Write down the extra conditions on U and f such that Assumption 4.9 (SLP, pp.84) holds. Show that these conditions imply the assumption. What can you say about v now?