

8106 Midterm Solution Sketch

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1) a) (5 pts)

$$\max_{\{c_t, n_t, l_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left(\theta \frac{c_t^{1-\sigma_1}}{1-\sigma_1} + (1-\theta) \frac{l_t^{1-\sigma_2}}{1-\sigma_2} \right)$$

$$\text{s.t. } c_t + k_{t+1} - (1-\delta)k_t \leq A k_t^\alpha n_t^{1-\alpha}$$

$$l_t + n_t \leq 1$$

$$c_t, n_t, l_t, k_{t+1} \geq 0$$

k₀ given

b) (15 pts)

λ_t Lagrange multiplier on feasibility

FOCs

$$[c_t]: \beta^t \theta c_t^{-\sigma_1} = \lambda_t$$

$$[l_t]: \beta^t (1-\theta) l_t^{-\sigma_2} = \lambda_t (1-\alpha) A k_t^{\alpha} n_t^{\alpha-1}$$

$$[k_{t+1}]: -\lambda_t + (1-\delta)\lambda_{t+1} + \lambda_{t+1} \alpha A k_{t+1}^{\alpha-1} n_{t+1}^{\alpha-1} = 0$$

$$\Rightarrow \frac{\lambda_t}{\lambda_{t+1}} = (1-\delta) + \alpha A k_{t+1}^{\alpha-1} n_{t+1}^{\alpha-1}$$

$$\text{Euler: } \frac{C_t^{-\sigma_1}}{C_{t+1}^{-\sigma_1}} = \beta \frac{\lambda_t}{\lambda_{t+1}} = \beta [\alpha A k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha} + (1-\delta)]$$

↳ MRS between consumption today and tomorrow is equal to discounted marginal change in output tomorrow. I.e. At optimum, cost of saving more today is equal to present value of benefit tomorrow (b/c of higher consumption tomorrow b/c of higher output).
See page 54 of Professor Kuegel's notes.

$$\text{Intra MRS: } \frac{(1-\theta) l_t^{-\sigma_2}}{\theta C_t^{-\sigma_1}} = (1-\alpha) A k_t^\alpha n_t^{1-\alpha}$$

↳ MRS between leisure and consumption at t is equal to marginal product of labor (i.e. wage rate). I.e. at optimum, cost of working more today equals benefit (through $\uparrow$ consumption today b/c of $\uparrow$ output).

$$\text{TVC: } \lim_{t \rightarrow \infty} \lambda_t k_{t+1} = 0$$

$$\text{or } \lim_{t \rightarrow \infty} \beta^t C_t^{-\sigma} [\alpha A k_t^{\alpha-1} n_t^{1-\alpha} + (1-\delta)] k_t = 0$$

↳ Terminal condition on capital stock.

Guarantees value (in terms of discounted marginal utility) of capital stock goes to zero as $t \rightarrow \infty$.
Ensures agent doesn't save "too much."

c) (5 pts)

A steady state is an optimal / equilibrium allocation
 $\{ \hat{c}_t, \hat{k}_t, \hat{n}_t, \hat{l}_t \}_{t=0}^{\infty}$ s.t. all quantities are
constant $\forall t \geq 0 : \hat{c}_t = \hat{c}^*, \hat{k}_t = \hat{k}^*, \hat{n}_t = \hat{n}^*, \hat{l}_t = \hat{l}^* \quad \forall t \geq 0$

Euler can be used to characterize the steady state value of capital stock:

$$\frac{c^{\alpha-\sigma}}{c^{\alpha-\sigma}} = 1 = \beta [A \alpha k^{\alpha-1} + (1-\delta)]$$
$$\Rightarrow k^* = \left(\frac{\frac{1}{\beta} - (1-\delta)}{A \alpha} \right)^{\frac{1}{\alpha-1}} = \left(\frac{A \alpha \beta}{1 - \beta(1-\delta)} \right)^{\frac{1}{1-\alpha}}$$

$$c^* = A k^{*\alpha} + (1-\delta)k^* - k^* = A k^{*\alpha} - \delta k^* \text{ from feasibility}$$

$$l_t^* = 0$$

$$n_t^* = 1$$

While we often know that the quantities converge to their steady state values, a steady state occurs only when the initial conditions begin at their steady state value (i.e. $k_0 = k^*$). Then all quantities will remain at their steady state values forever.

d) (5 pts)

$$V(k) = \max_{c, l, n, k'} \left\{ \theta \frac{c^{1-\sigma_1}}{1-\sigma_1} + (1-\theta) \frac{l^{1-\sigma_2}}{1-\sigma_2} + \beta V(k') \right\}$$

$$\text{s.t.} \quad c + k' - (1-\delta)k \leq Ak^\alpha n^{1-\alpha}$$

$$l + n \leq 1$$

$$c, k', n, l \geq 0$$

State variable: k

Control variables: c, l, n, k'

2) a) (5 pts)

$$v(k) = \max_{k'} \left\{ \frac{(Ak - k')^{1-\sigma}}{1-\sigma} + \beta v(k') \right\}$$

$$\text{s.t. } k' \in \{k' \in K \mid k' \in [0, Ak]\} \equiv \Gamma(k)$$

x is k

y is k'

$$X \equiv K = [0, +\infty)$$

↳ No maximum sustainable capital as only $\tilde{k}=0$ solves $\tilde{k} = A\tilde{k}$

$$F(k, k') = \frac{(Ak - k')^{1-\sigma}}{1-\sigma}$$

$$= \max_c \frac{c^{1-\sigma}}{1-\sigma}$$

$$\text{s.t. } c + k' \leq Ak$$

$$c \geq 0$$

k, k' fixed

$$\Gamma(k) = \{k' \in K \mid k' \in [0, Ak]\}$$

b) (5 pts)

$$\underline{\text{FOC}}: (Ak - k')^{-\sigma} = \beta v'(k')$$

$$\underline{\text{ENV}}: A(Ak - k')^{-\sigma} = v'(k)$$

c) (15 pts)

$$a_1 \frac{k'^{-\sigma}}{1-\sigma} = v(k) = \max_{k'} \left\{ \frac{(Ak - k')^{1-\sigma}}{1-\sigma} + \beta a_1 \frac{k'^{1-\sigma}}{1-\sigma} \right\}$$

FOC:

$$[k']: -(Ak - k')^{-\sigma} + \beta a_1 k'^{-\sigma} = 0$$

$$\Rightarrow \beta a_1 k'^{-\sigma} = (Ak - k')^{-\sigma}$$

$$\Rightarrow Ak - k' = (\beta a_1)^{-\frac{1}{\sigma}} k'$$

$$\Rightarrow k'(1 + (\beta a_1)^{-\frac{1}{\sigma}}) = Ak$$

$$\Rightarrow g_k(k) = k' = \frac{Ak}{(1 + (\beta a_1)^{-\frac{1}{\sigma}})}$$

$$= \frac{Ak}{\left(\frac{(\beta a_1)^{\frac{1}{\sigma}}}{(\beta a_1)^{\frac{1}{\sigma}}} + \frac{1}{(\beta a_1)^{\frac{1}{\sigma}}} \right)} = \frac{(\beta a_1)^{\frac{1}{\sigma}} Ak}{1 + (\beta a_1)^{\frac{1}{\sigma}}}$$

$$g_c(k) = Ak - g(k)$$

Method 1: Plug k' back into value function & solve for a_1 :

$$\text{Let } (1 + (\beta a_1)^{-\frac{1}{\sigma}}) = \Theta \Leftrightarrow \frac{1}{\Theta} = \frac{1}{(1 + (\beta a_1)^{-\frac{1}{\sigma}})} = \frac{(\beta a_1)^{\frac{1}{\sigma}}}{1 + (\beta a_1)^{\frac{1}{\sigma}}}$$

$$\frac{a_1 k^{1-\sigma}}{1-\sigma} = \frac{(Ak - Ak/\Theta)^{1-\sigma}}{1-\sigma} + \beta a_1 \frac{(Ak/\Theta)^{1-\sigma}}{1-\sigma}$$

$$= \frac{(Ak(1 - \frac{1}{\Theta}))^{1-\sigma}}{1-\sigma} + \frac{\beta a_1}{\Theta^{1-\sigma}} \frac{(Ak)^{1-\sigma}}{1-\sigma}$$

$$\cancel{\frac{a_1 k^{1-\sigma}}{1-\sigma}} = \cancel{\frac{k^{1-\sigma}}{1-\sigma}} \left(A^{1-\sigma} \left(1 - \frac{1}{\Theta}\right)^{1-\sigma} + \frac{\beta a_1 A^{1-\sigma}}{\Theta^{1-\sigma}} \right)$$

$$\Rightarrow a_1 = \left(A^{1-\sigma} \left(1 - \frac{(\beta a_1)^{\frac{1}{\sigma}}}{1 + (\beta a_1)^{\frac{1}{\sigma}}}\right)^{1-\sigma} + \beta a_1 \left(\frac{(\beta a_1)^{\frac{1}{\sigma}} A}{1 + (\beta a_1)^{\frac{1}{\sigma}}} \right)^{1-\sigma} \right)$$

$$\Rightarrow a_1 = \left(\frac{A^{1-\sigma}}{(1 + (\beta a_1)^{\frac{1}{\sigma}})^{1-\sigma}} + \beta a_1 \frac{(\beta a_1)^{\frac{1-\sigma}{\sigma}} A^{1-\sigma}}{(1 + (\beta a_1)^{\frac{1}{\sigma}})^{1-\sigma}} \right)$$

$$\Rightarrow a_1 = \left(\frac{A^{1-\sigma}}{(1 + (\beta a_1)^{\frac{1}{\sigma}})^{1-\sigma}} + \frac{(\beta a_1)^{\frac{1}{\sigma}} A^{1-\sigma}}{(1 + (\beta a_1)^{\frac{1}{\sigma}})^{1-\sigma}} \right)$$

$$= A^{1-\sigma} \left(\frac{1 + (\beta a_1)^{\frac{1}{\sigma}}}{(1 + (\beta a_1)^{\frac{1}{\sigma}})^{1-\sigma}} \right)$$

$$a_1 = A^{1-\sigma} (1 + (\beta a_1)^{1/\sigma})^\sigma$$

$$\Rightarrow a_1^{1/\sigma} = A^{1-\sigma} + A^{1-\sigma} (\beta a_1)^{1/\sigma}$$

$$\Rightarrow a_1^{1/\sigma} (1 - A^{1-\sigma} \beta^{1/\sigma}) = A^{1-\sigma}$$

$$\Rightarrow a_1 = \left(\frac{A^{1-\sigma}}{1 - A^{1-\sigma} \beta^{1/\sigma}} \right)^\sigma$$

Method 2: Set policy rules from FOC + ENV equal:

$$\text{ENV: } \frac{a_1}{k^\sigma} = \frac{A}{(Ak - k')^\sigma}$$

$$\Leftrightarrow a_1 (Ak - k')^\sigma = Ak^\sigma$$

$$\Leftrightarrow Ak - k' = \left(\frac{A}{a_1} \right)^{1/\sigma} k$$

$$\Leftrightarrow g_k(k) = k' = k \left(A - \left(\frac{A}{a_1} \right)^{1/\sigma} \right)$$

Set $g_k(k)$ from FOC equal to this:

$$\frac{(\beta a_1)^{1/\sigma} Ak}{1 + (\beta a_1)^{1/\sigma}} = k \left(A - \left(\frac{A}{a_1} \right)^{1/\sigma} \right)$$

Solve for a_1 .

Quickest
option

Method 3: Use FOC and ENV to directly obtain policy function. Then set equal to FOC/ENV policy function and solve for a_1 .

$$\text{ENV: } v'(k) = A(Ak - k')^{-\sigma}$$

$$\text{FOC: } \beta v'(k') = (Ak - k')^{-\sigma}$$

Divide ENV by FOC:

$$\frac{v'(k)}{\beta v'(k')} = A$$

$$\text{Use } v(k) = \frac{a_1 k^{1-\sigma}}{1-\sigma}$$

$$\Rightarrow \frac{v'(k)}{\beta v'(k')} = \frac{1}{\beta} \left(\frac{k}{k'} \right)^{-\sigma} = A$$

$$\Rightarrow \left(\frac{k'}{k} \right)^{\sigma} = A\beta$$

$$\Rightarrow g_k(k) = k' = (A\beta)^{\frac{1}{\sigma}} k$$

The policy rule is linear with slope $(A\beta)^{\frac{1}{\sigma}}$.

Set equal to ENV policy rule:

$$\cancel{k} (A\beta)^{\frac{1}{\sigma}} = \cancel{k} \left(A - \left(\frac{A}{a_1} \right)^{\frac{1}{\sigma}} \right)$$

$$\Rightarrow a_1^{\frac{1}{\sigma}} (A\beta)^{\frac{1}{\sigma}} = a_1^{\frac{1}{\sigma}} A - A^{\frac{1}{\sigma}}$$

$$\Rightarrow a_1''^\sigma A (1 - A^{\frac{1}{\sigma}-1} \beta^{\frac{1}{\sigma}}) = A^{\frac{1}{\sigma}}$$

$$\begin{aligned} \Rightarrow a_1^{\frac{1}{\sigma}} &= \frac{A^{\frac{1}{\sigma}}}{A (1 - A^{\frac{1}{\sigma}} \beta^{\frac{1}{\sigma}})} \\ &= \frac{A^{\frac{1}{\sigma}-1}}{(1 - A^{\frac{1}{\sigma}} \beta^{\frac{1}{\sigma}})} \end{aligned}$$

$$\Rightarrow a_1^{\frac{1}{\sigma}} = \frac{A^{\frac{1}{\sigma}}}{1 - A^{\frac{1}{\sigma}} \beta^{\frac{1}{\sigma}}}$$

$$a_1 = \left(\frac{A^{\frac{1}{\sigma}}}{1 - A^{\frac{1}{\sigma}} \beta^{\frac{1}{\sigma}}} \right)^\sigma$$

Which is the same as earlier.

d) (EC - 5 pts)

If agent never consumes; then $k_t = A^t k_0 \quad \forall t$

$$\Rightarrow v(k_t) = \frac{a_1 (A^t k_0)^{1-\sigma}}{1-\sigma}$$

$$\Rightarrow \beta^t v(k_t) = \beta^t \frac{a_1 (A^t k_0)^{1-\sigma}}{1-\sigma}$$

$$= a_1 \beta^t A^{t(1-\sigma)} \frac{k_0^{1-\sigma}}{1-\sigma}$$

$$= a_1 (\underbrace{\beta A^{1-\sigma}}_{\in (0,1)})^t \frac{k_0^{1-\sigma}}{1-\sigma}$$

$$\Rightarrow \beta^t v(k_t) \rightarrow 0 \quad \text{as } t \rightarrow \infty \quad \forall k_0 \in K$$

What if never saves?

$$v(k_0) = a_1 \frac{k_0^{1-\sigma}}{1-\sigma}; \quad v(0) = 0$$

$$= \left\{ a_1 \frac{k_0^{1-\sigma}}{1-\sigma}, 0, 0, 0, \dots \right\}$$

Then for all other feasible paths, $k_t \in [0, A^t k_0)$, so that

$\beta^t v(k_t) \rightarrow 0$ for all feasible paths.

3) a) $x_t = d_t$
 $F(d_t, d_{t+1}) = u_1(w + (1-\delta)d_t - d_{t+1}) + u_2(d_t)$ } u_1, u_2 st. increasing

$$= \max_{c_t} u_1(c_t) + u_2(d_t)$$

s.t. $c_t + d_{t+1} - (1-\delta)d_t \leq w$
 $c_t \geq 0$
 d_t, d_{t+1} fixed

$$\Gamma(d_t) = \{d_{t+1} \in D \mid d_{t+1} \in [0, w + (1-\delta)d_t]\}$$

$$X \equiv D = [0, \frac{w}{\delta}] ; \left[\underbrace{w + (1-\delta)\tilde{d}}_{\text{Maximum sustainable durable good}} = \tilde{d} \Leftrightarrow \tilde{d} = \frac{w}{\delta} \right]$$

or $D = [0, +\infty]$

$$\Rightarrow \max_{\{d_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(d_t, d_{t+1})$$

s.t. $d_{t+1} \in \Gamma(d_t) \quad \forall t \geq 0$
 $d_0 \in D$ given

- $\sum_{t=0}^{\infty} \beta^t F(d_t, d_{t+1})$ exists $\forall$ feasible paths since $\beta \in (0, 1)$ and F is guaranteed to be bounded below (since F continuous).
 See pg. 69 of SLP. u_1, u_2 continuous
- $\Rightarrow$ A4.2 satisfied

- $\Gamma(d_t)$ nonempty $\forall d_t$ since always can chose $d_{t+1} = 0$:
 $w + (1-\delta)d_t > 0 \quad \forall d_t \in D$
 $\Rightarrow [0, w + (1-\delta)d_t]$ non-empty $\forall d_t \in D$
- $\Rightarrow$ A4.1 satisfied

b) (5 pts)

$$v(d) = \max_{d' \in \Gamma(d)} \left[\overbrace{u_1(w + (1-\sigma)d - d')}^{F(d,d')} + u_2(d) + \beta v(d') \right]$$

v^* satisfies FE since A4.1 and A4.2 are satisfied $\Rightarrow$ Theorem 4.2 holds.

c) (10 points) Want Theorem 4.6 to hold.

Need A4.4: F bounded and continuous, and $\beta \in (0,1)$

A4.3: D convex set of $\mathbb{R}$, and
 Γ is non-empty, compact valued and continuous.

If a) $D = [0, \frac{\omega}{\delta}]$, no more assumptions needed.

i) u_1 and u_2 continuous $\Rightarrow F$ continuous

ii) u_1 and u_2 bounded above and below
since continuous on compact set D
 $\Rightarrow F$ bounded

iii) $\beta \in (0,1)$ by assumption

We needed $\delta \in (0,1]$ for $\tilde{d} = \frac{\omega}{\delta}$ to exist.

Clearly $D \subseteq \mathbb{R}$ and convex. And Γ non-empty, compact, continuous.
 $\omega + (1-\delta)d > 0$

If b) $D = [0, +\infty)$, need assumption to ensure F bounded:

i) u_1 and u_2 continuous $\Rightarrow F$ continuous

ii) Assume u_1 and u_2 bounded above; already
bounded below since continuous on $[0, +\infty)$
 $\Rightarrow F$ bounded

iii) $\beta \in (0,1)$ by assumption $\rightarrow$ I.e. u_1, u_2 have horizontal asymptote

Clearly $D \subseteq \mathbb{R}$ and convex. And Γ non-empty, compact, continuous.
 $\omega + (1-\delta)d > 0$

Note $\omega > 0$ is a constant. You do not have to make assumptions about a production function.

c) (10 pts) Want Theorem 4.11 to hold.

Need A4.3-A4.4 $\Rightarrow$ Same assumptions as (b)

A4.7: F strictly concave

A4.8: Γ is convex

A4.9: F is continuously differentiable

} Theorem 4.8
holds
 $\Rightarrow v$ st.
concave

Add i) u_1 and u_2 strictly concave $\Rightarrow F$ strictly concave

ii) u_1 and u_2 continuously diff $\Rightarrow F$ continuously diff.

Didn't
need to
prove this }

Γ is also clearly convex. Take $d_1, d_2 \in D$:

$$d_1' \in [0, w + (1-\sigma)d_1] \quad \text{and} \quad d_2' \in [0, w + (1-\sigma)d_2]$$

$$\Rightarrow 0 \leq \theta_i d_i' \leq \theta_i w + (1-\sigma)\theta_i d_i \quad \text{for } i \in \{1, 2\}; \theta_1 + \theta_2 = 1; \theta_i \in [0, 1]$$

$$\Rightarrow 0 \leq \theta_1 d_1' + \theta_2 d_2' \leq w + (1-\sigma)(\theta_1 d_1 + \theta_2 d_2)$$

$$\Rightarrow \theta_1 d_1' + \theta_2 d_2' \in \Gamma(\theta_1 d_1 + \theta_2 d_2)$$

$$\text{FOC: } u_1'(w + (1-\sigma)d - d') = \beta v'(d')$$

$$\text{ENV: } v'(d) = (1-\sigma) u_1'(w + (1-\sigma)d - d') + u_2'(d)$$

e) (10 pts) Proof by contradiction.

Suppose $d_1 > d_2$ but $g(d_1) \leq g(d_2)$

$$w + (1-\delta)d_1 - g(d_1) > w + (1-\delta)d_2 - g(d_2) \quad \text{since } d_1 > d_2 \text{ and } -g(d_1) \geq -g(d_2) \text{ and } \delta \in [0,1].$$

$$u'_1(w + (1-\delta)d_1 - g(d_1)) < u'_1(w + (1-\delta)d_2 - g(d_2))$$

by st. concavity of $u_1 \Rightarrow u'_1$ strictly decreasing.

$$\beta v'(g(d_1)) < \beta v'(g(d_2)) \quad \text{from FOC}$$

$$g(d_1) > g(d_2) \quad \text{by st. concavity of } v. \quad \text{Theorem 4.8}$$

Contradiction. Assumed $g(d_1) \leq g(d_2)$ but showed $g(d_1) > g(d_2)$.

Thus policy function for durable goods is strictly increasing.