

Macroeconomic Theory (Econ 8106)

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Midterm Exam

Please write your answers in the provided booklet. You have one hour and 30 minutes to work on this exam. Show your work for full credit.

Question 1: (30 points)

Consider an economy in which the representative consumer lives forever. There is a good in each period that can be consumed or saved as capital as well as labor. The consumer's utility function is

$$\sum_{t=0}^{\infty} \beta^t \left(\theta \frac{c_t^{1-\sigma_1}}{1-\sigma_1} + (1-\theta) \frac{l_t^{1-\sigma_2}}{1-\sigma_2} \right)$$

where $\beta \in (0, 1)$, $\theta \in (0, 1)$. Also assume $\sigma_1, \sigma_2 \in (0, \infty)$ but that $\sigma_1, \sigma_2 \neq 1$. The consumer is endowed with 1 unit of time each period, some of which can be consumed as leisure l_t , and some of which is supplied as labor n_t . The consumer is also endowed with $k_0 > 0$ units of capital in period 0. Feasible allocations satisfy:

$$c_t + k_{t+1} - (1 - \delta)k_t \leq Ak_t^\alpha n_t^{1-\alpha}$$

where $A > 0$, $\alpha \in (0, 1)$ and $\delta \in (0, 1]$.

- (a) (5 points) Write down the social planner's problem.
- (b) (15 points) Write down the Euler equation for consumption, the intratemporal marginal rate of substitution between consumption and labor, and the transversality condition. Provide a short interpretation of each of these three conditions.
- (c) (5 points) Define and characterize a steady state for this economy. You can assume that labor is supplied inelastically (e.g. $\theta = 1$) when characterizing the steady state.
- (d) (5 points) Formulate the appropriate Bellman equation in this setup (you do not have to write it in SLP's canonical form). What is the state variable and what are the control variable(s) in this problem?

Question 2: (25 points)

We will simplify the previous model. Assume that there is no endowment of time (no labor/leisure choice). The consumer's utility function is:

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

where $\beta \in (0, 1)$. Also assume $\sigma \in (0, \infty)$ but that $\sigma \neq 1$. The consumer is endowed with $k_0 > 0$ units of capital in period 0. Feasible allocations satisfy:

$$c_t + k_{t+1} \leq A k_t$$

where $A > 0$.

- (a) (5 points) Write down the Bellman equation for this problem using SLP's canonical form/notation:

$$v(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta v(y), \quad \forall x \in X$$

Make sure you clearly define what x and y are in this problem. Also define the set X of possible values for the state variable, the constraint correspondence Γ , and the one period return function F .

- (b) (5 points) Characterize the optimal policy through the first order and envelope condition.
- (c) (15 points) Guess that the value function takes the form

$$v(k) = a_1 \frac{k^{1-\sigma}}{1-\sigma}$$

where a_1 is a constant.

Use this to solve for a_1 and the policy functions $g_k(k)$ and $g_c(k)$ (once you solve for a_1 , you do not have to plug the formula for a_1 back into the policy functions).

- (d) (**Extra Credit:** 5 points) Assume that $0 < A < (\frac{1}{\beta})^{\frac{1}{1-\sigma}}$ (so $0 < A^{(1-\sigma)}\beta < 1$) and that $0 < \sigma < 1$. Given these assumptions and $v(k)$ from part (c), show that

$$\lim_{t \rightarrow \infty} \beta^t v(k_t) = 0$$

for **all feasible** paths of capital; i.e. all $\{k_t\}_{t=0}^{\infty}$ such that $k_{t+1} \in \Gamma(k_t) \quad \forall t$ and $k_0 \in K$. Thus by SLP Theorem 4.3 we know that the value function (v) and supremum function (v^* from the associated sequence problem) are equivalent ($v = v^*$). This setup also implies that SLP Theorem 4.5 holds; which tells us that any plan generated by the policy function is equivalent to the optimal plan from the sequence problem.

Hint: What happens to the sequence $\{\beta^0 v(k_0), \beta^1 v(k_1), \beta^2 v(k_2), \dots, \beta^t v(k_t), \dots\}$ if the agent never consumes? How about if they never save?

Question 3: (45 points)

Consider a single agent problem where each period, $w > 0$ total output is produced and can be divided into consumption of a perishable good, c_t , and into a durable good, d_{t+1} . The durable depreciates like a capital good, but is not directly productive. The stock of durables at any date, d_t , produces a flow of services that enters the utility function.

Thus, the problem faced by the household with initial stock d_0 is:

$$\begin{aligned} \max_{\{c_t, d_{t+1}\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t [u_1(c_t) + u_2(d_t)] \\ \text{s.t. } & c_t + d_{t+1} - (1 - \delta)d_t \leq w \quad \forall t \\ & c_t, d_{t+1} \geq 0 \quad \forall t \\ & d_0 \text{ given} \end{aligned}$$

where both u_1 and u_2 are strictly increasing and continuous on $[0, +\infty)$. Also assume $\beta \in (0, 1)$ and $\delta \in (0, 1]$.

- (a) (10 points) Write the above problem in SLP's canonical sequence problem form:

$$\begin{aligned} \max_{\{x_{t+1}\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \\ \text{s.t. } & x_{t+1} \in \Gamma(x_t) \quad \forall t \\ & x_0 \in X \text{ given} \end{aligned}$$

Make sure you clearly define the state variable x_t , the constraint correspondence Γ , the set X of possible values for the state variable, and the one period return function F . Given the current set of assumptions, briefly explain how we know that $\sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$ exists for all feasible sequences $\{x_t\}_{t=0}^{\infty}$ (although this sum may be $-\infty$ or $+\infty$). Briefly explain why $\Gamma(x_t)$ non-empty for all $x_t \in X$.

- (b) (5 points) Write the Bellman/functional equation for this problem in SLP's canonical form:

$$v(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta v(y)$$

Briefly explain how we know that the supremum function from the sequence problem in (a) satisfies this functional equation.

- (c) (10 points) Provide assumptions on u_1 and u_2 that are sufficient to guarantee that there is a unique bounded and continuous function satisfying the functional equation in (b). Briefly explain why each assumption is needed.
- (d) (10 points) Make appropriate assumptions on u_1 and u_2 to characterize the optimal policy function through the first-order and envelope condition, and derive the conditions (you can assume that the optimal quantities are strictly positive when deriving the first order and envelope condition; e.g. u_1 and u_2 satisfy the Inada conditions).
- (e) (10 points) Use the conditions in (c) to show that the optimal policy function for durable goods is strictly increasing. Clearly explain each step.