

2) a) TDCF is:

$$i) \hat{z}^H \equiv \{\hat{c}_t, \hat{l}_t, \hat{n}_t, \hat{k}_{t+1}, \hat{x}_t\}_{t=0}^{\infty}$$

$$ii) \hat{z}^F \equiv \{\hat{y}_t^f, \hat{n}_t^f, \hat{k}_t^f\}_{t=0}^{\infty}$$

$$iii) \hat{p} \equiv \{\hat{p}_t, \hat{r}_t, \hat{w}_t\}_{t=0}^{\infty}$$

$$iv) \hat{G} \equiv \{g_t, \tau_{ct}, \tau_{xt}, \tau_k\}_{t=0}^{\infty}, \text{ } k_0 \text{ given such that}$$

1) Given $\hat{p}$ and $\hat{G}$, the consumer chooses $\hat{z}^H$ to solve:

$$\max_{\hat{z}^H} \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

$$\text{s.t.} \quad \sum_{t=0}^{\infty} \hat{p}_t (1 + \tau_{ct}) c_t + \hat{p}_t (1 + \tau_{xt}) x_t$$

$$\leq \sum_{t=0}^{\infty} (1 - \tau_k) [n_t \hat{w}_t + \hat{r}_t^k k_t]$$

$$k_{t+1} \leq x_t + (1 - d) k_t$$

$$n_t + l_t \leq 1 \quad \forall t$$

$$c_t, k_{t+1}, n_t, l_t \geq 0 \quad \forall t$$

2) Given $\hat{p}$ and $\hat{G}$, k_0 given, the firm chooses $\hat{z}^F$ to max profit:

$$\max_{\hat{z}^F} \sum_{t=0}^{\infty} \hat{p}_t y_t^f - \hat{w}_t n_t^f - \hat{r}_t^k k_t^f$$

$$\text{s.t.} \quad y_t^f \leq F(k_t^f, n_t^f)$$

$$y_t^f, n_t^f, k_t^f \geq 0$$

3) Markets clear $\forall t$

$$\hat{c}_t + \hat{x}_t + g_t = F(\hat{k}_t^f, \hat{n}_t^f) = \hat{y}_t^f$$

$$\hat{n}_t^f = \hat{n}_t$$

$$\hat{k}_t^f = \hat{k}_t$$

4) Government Budget balance each period:

$$\hat{p}_t g_t = \tau_c \hat{p}_t \hat{C}_t + \tau_x \hat{p}_t \hat{x}_t + \tau [\bar{w}_t \hat{n}_t + \hat{r}_t^k \hat{k}_t] \quad \forall t$$

b)

FOCs: λ on BC. Plug in law of motion of capital

$$[C_t]: \beta^t u_c(t) = \lambda \hat{p}_t (1 + r_{c1})$$

$$[l_t]: \beta^t u_l(t) = \lambda (1 - \tau) \hat{w}_t$$

$$[k_{t+1}]: -\lambda (1 + r_{x1}) p_t + \lambda (1 - \delta) p_{t+1} (1 + r_{x+1}) + \lambda r_{t+1}^k (1 - \tau) = 0$$

$$\Rightarrow p_t = \frac{r_{t+1} (1 - \tau)}{(1 + r_{x1})} + p_{t+1} \frac{(1 + r_{x+1}) (1 - \delta)}{(1 + r_{x1})}$$

$$\Leftrightarrow \frac{p_t}{p_{t+1}} = \frac{r_{t+1}}{p_{t+1}} \frac{(1 - \tau)}{(1 + r_{x1})} + \frac{(1 + r_{x+1}) (1 - \delta)}{(1 + r_{x1})}$$

Firm FOC

$$\hat{r}_t^k = \hat{p}_t F_k(t)$$

$$\hat{w}_t = \hat{p}_t F_n(t)$$

$$p_0 = 1$$

$$\hat{p}_t = \beta^t \frac{u_c(t)}{u_c(0)} \frac{(1+r_{c0})}{(1+r_{ct})}$$

$$EE: \frac{u_c(t)}{\beta u_c(t+1)} = \frac{p_t (1+r_{ct})}{p_{t+1} (1+r_{c,t+1})}$$

$$\frac{p_t}{p_{t+1}} = \left[\frac{F_k(t+1)(1-\tau_t)}{(1+r_{xt})} + \frac{(1-\delta)(1+r_{x,t+1})}{(1+r_{xt})} \right]$$

$$IMRS: \frac{u_c(t)}{u_x(t)} = \frac{p_t (1+r_{ct})}{w_t (1-\tau_t)} = \frac{(1+r_{ct})}{F_n(t)(1-\tau_t)}$$

$$TVC: \lim_{t \rightarrow \infty} p_t k_{t+1} = 0$$

$$\text{Initial: } k_0 > 0 \text{ given}$$

$$\text{Firm: } r_t^k = p_t F_k(t) \\ w_t = p_t F_n(t)$$

$$\text{Gov: } p_t g_t = \tau_{ct} p_t C_t + \tau_{xt} p_t X_t + \tau_t [p_t F_n(t) n_t + p_t F_k(t) k_t] \\ = p_t [\tau_{ct} C_t + \tau_{xt} X_t + \tau_t \underbrace{[F_n(t) n_t + F_k(t) k_t]}_{F(k_t, n_t)}]$$

$$\Leftrightarrow g_t = \tau_{ct} C_t + \tau_{xt} X_t + \tau_t F(k_t, n_t)$$

This is a key part for defining feasibility constraint in the SPP.

$$MC: C_t + X_t + g_t = F(k_t, n_t)$$

$$\text{law motion capital: } k_{t+1} \leq (1-\delta) k_t + X_t$$

↓
Plug this into market clearing:

$$C_t + X_t + \tau_{ct} C_t + \tau_{xt} X_t$$

$$= F(k_t, n_t) - \tau_t F(k_t, n_t)$$

$$\Rightarrow (1 + \tau_{ct}) C_t + (1 + \tau_{xt}) X_t = (1 - \tau_t) F(k_t, n_t)$$

c) SPP: Let $a_t = (1 + r_{ct})$

$$b_t = (1 + r_{xt})$$

$$\hat{F}_t(k_t, n_t) = (1 - \tau_t) F(k_t, n_t)$$

$$\max_{\{c_t, x_t, k_{t+1}, n_t, l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

s.t. $a_t c_t + b_t x_t = \hat{F}_t(k_t, n_t)$

$$k_{t+1} \leq x_t + (1 - \delta) k_t$$

$$n_{t+1} + l_t \leq 1$$

$$c_t, n_t, l_t, k_t \geq 0$$

k_0 given

d) FOCs: μ_t on feasibility

$$[c_t]: \beta^t u_c(t) = \mu_t a_t$$

$$[l_t]: \beta^t u_l(t) = \mu_t \hat{F}_n(t)$$

$$[k_{t+1}]: -\lambda_t b_t + \lambda_{t+1} (1 - \delta) b_{t+1} + \lambda_{t+1} \hat{F}_k(t+1) = 0$$

$$\Rightarrow \lambda_t = \lambda_{t+1} (1 - \delta) \frac{b_{t+1}}{b_t} + \frac{\lambda_{t+1}}{b_t} \hat{F}_k(t+1)$$

$$\Rightarrow \frac{\lambda_t}{\lambda_{t+1}} = (1 - \delta) \frac{b_{t+1}}{b_t} + \frac{\hat{F}_k(t+1)}{b_t}$$

$$EF: \frac{u_c(t)}{\beta u_c(t+1)} = \frac{\lambda_t}{\lambda_{t+1}} \frac{a_t}{a_{t+1}}$$

$$\begin{aligned} \frac{\lambda_t}{\lambda_{t+1}} &= \left[\frac{\hat{F}_k(t+1)}{b_t} + (1-d) \frac{b_{t+1}}{b_t} \right] \\ &= \frac{(1+r_{ct})}{(1+r_{ct+1})} \left[\frac{(1-r_t) F_k(t)}{(1+r_{kt})} + (1-d) \frac{(1+r_{xt+1})}{(1+r_{xt})} \right] \end{aligned}$$

$$IMRS: \frac{u_c(t)}{u_x(t)} = \frac{a_t}{\hat{F}_n(t)} = \frac{(1+r_{ct})}{(1-r_t) F_n(t)}$$

$$TVC: \lim_{t \rightarrow \infty} M + k_{t+1} = 0$$

$$Fcos: a_t c_t + b_t x_t = \hat{F}_t(k_t, n_t)$$

$$\Rightarrow (1+r_{ct})c_t + (1+r_{xt})x_t = (1-r_t) F(k_t, n_t)$$

$$\Rightarrow c_t + x_t + \underbrace{r_{ct}c_t + r_{xt}x_t + r_t F(k_t, n_t)}_{= g_t \text{ from above}} = F(k_t, n_t)$$

$$\text{Law of motion: } k_{t+1} \leq x_t + (1-d)k_t$$

These are same equations as in (b).

$$\begin{aligned} e) \quad v(k) &= \max_{c, l, n, x, k'} u(c, l) + \beta v(k') \\ \text{s.t.} \quad & ac + bx \leq \hat{F}(k, n) \\ & n + l \leq 1 \\ & k' \leq (1-d)k + x \\ & c, n, l, k' \geq 0 \end{aligned}$$

You could pretty easily code this!