

Recitation 7 - 8105

Zach Swaziek

October 18, 2022

1 Overview

In these notes,¹ we generalize the infinitely lived model with a single household and allow there to be a N households. We define ADE and SME for this case (this part is identical to the recitation 5 notes). We also define Pareto efficiency, the social planner's problem (SPP), and write down the Bellman equation for the SPP.

At the end of these notes, we also define ADE and SME for a two-period OLG model with production and N types of consumers.

2 ADE/SME with Production and N Infinitely Lived Agents

In recitation, we discussed the case when labor was supplied inelastically² and there were only two households ($N = 2$). I write the general case here. Note that since we have N households, we set $I \equiv \{1, 2, \dots, N\}$.

We assume the following endowments:

- Time endowments³ of $\bar{\ell}_t^i \forall i \in I, \forall t$
- Initial capital stock $\bar{k}_0^i > 0 \forall i \in I$
- (If in SME) Initial bond $\bar{b}_0^i = 0 \forall i \in I$

Be careful with notation. The notation I use here is different compared to what we used in recitation. In these notes, n is labor and l is leisure.

¹These notes are adapted from Professor Tim Kehoe's lectures, Alex Wurdinger's Fall 2022 recitations, Professor Larry Jones' notes, and past Econ 8105 TAs' (Conor Ryan and Monica Tran Xuan) teaching notes. I do not claim credit for these notes and errors are my own.

²Since leisure was not in the utility function, the households supplied all of their endowment of labor.

³The households choose between spending time on labor (n) or leisure (l). The sum of time spent on labor and leisure for household i in period t cannot exceed $\bar{\ell}_t^i$ (i.e. $n_t^i + l_t^i \leq \bar{\ell}_t^i$).

On the next two pages, I will define an ADE and SME with production and N households.

If you set $N = 1$, you will get exactly the same ADE/SME definitions as I defined in the previous set of notes (in the single agent infinitely lived model). If you set $N = 2$, the equilibrium definitions will be identical to those I discussed during recitation 5 (other than the labor supply choice).

Definition: An **ADE with production and N households** consists of:

- i.) Household Allocations: $\hat{Z}^{Hi} \equiv \{\hat{c}_t^i, \hat{k}_{t+1}^i, \hat{n}_t^i, \hat{l}_t^i\}_{t=0}^\infty \quad \forall i \in I$
 - ii.) Firm Production Plan: $\hat{Z}^F \equiv \{\hat{y}_t^f, \hat{k}_t^f, \hat{n}_t^f\}_{t=0}^\infty$
 - iii.) Price System: $\hat{P} \equiv \{\hat{p}_t, \hat{r}_t^k, \hat{w}_t\}_{t=0}^\infty$
- with $\hat{k}_0^i = \bar{k}_0^i \quad \forall i \in I$ given, such that:

- 1.) Given $\hat{P}$, household $i \in I$ chooses $\hat{Z}^{Hi}$ to solve:

$$\begin{aligned} & \max_{\{c_t^i, k_{t+1}^i, n_t^i, l_t^i\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t u_i(c_t^i, l_t^i) \\ \text{s.t.} \quad & \sum_{t=0}^\infty \hat{p}_t (c_t^i + k_{t+1}^i - (1 - \delta)k_t^i) \leq \sum_{t=0}^\infty \hat{w}_t n_t^i + \hat{r}_t^k k_t^i \\ & n_t^i + l_t^i \leq \bar{l}_t^i \quad \forall t \\ & c_t^i, k_{t+1}^i, l_t^i, n_t^i \geq 0 \quad \forall t \\ & \hat{k}_0^i = \bar{k}_0^i \quad \text{given} \end{aligned}$$

- 2.) Prices are s.t. the firm chooses $\hat{Z}^F$ to minimize costs and make zero profits:

$$\begin{aligned} \hat{y}_t^f &= F(\hat{k}_t^f, \hat{n}_t^f) \quad \forall t \\ \hat{r}_t^k &= \hat{p}_t F_k(\hat{k}_t^f, \hat{n}_t^f) \quad \forall t \\ \hat{w}_t &= \hat{p}_t F_n(\hat{k}_t^f, \hat{n}_t^f) \quad \forall t \end{aligned}$$

- 3.) Markets clear in each period:

$$\left(\sum_{i=1}^N \hat{c}_t^i \right) + \hat{k}_{t+1}^f - (1 - \delta)\hat{k}_t^f = F(\hat{k}_t^f, \hat{n}_t^f) = \hat{y}_t^f \quad \forall t \quad (\text{Goods Market})$$

$$\sum_{i=1}^N \hat{k}_t^i = \hat{k}_t^f \quad \forall t \quad (\text{Capital Market})$$

$$\sum_{i=1}^N \hat{n}_t^i = \hat{n}_t^f \quad \forall t \quad (\text{Labor Market})$$

Definition: A SME with production and N households consists of:

- i.) Household Allocations: $\hat{Z}^{Hi} \equiv \{\hat{c}_t^i, \hat{k}_{t+1}^i, \hat{n}_t^i, \hat{l}_t^i, \hat{b}_{t+1}^i\}_{t=0}^\infty \quad \forall i \in I$
 - ii.) Firm Production Plan: $\hat{Z}^F \equiv \{\hat{y}_t^f, \hat{k}_t^f, \hat{n}_t^f\}_{t=0}^\infty$
 - iii.) Price System:⁴ $\tilde{P} \equiv \{\tilde{r}_t^k, \tilde{w}_t, \tilde{r}_t^b\}_{t=0}^\infty$
- with $\hat{k}_0^i = \bar{k}_0^i \quad \forall i \in I$ and $\hat{b}_0^i = \bar{b}_0^i = 0 \quad \forall i \in I$ given, such that:

- 1.) Given $\tilde{P}$, household $i \in I$ chooses $\hat{Z}^{Hi}$ to solve:

$$\begin{aligned} & \max_{\{c_t^i, k_{t+1}^i, n_t^i, l_t^i, b_{t+1}^i\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t u_i(c_t^i, l_t^i) \\ \text{s.t.} \quad & c_t^i + k_{t+1}^i - (1 - \delta)k_t^i + b_{t+1}^i \leq \tilde{w}_t n_t^i + \tilde{r}_t^k k_t^i + (1 + \tilde{r}_t^b) b_t^i \quad \forall t \\ & b_t^i \geq -B \quad \forall t \\ & n_t^i + l_t^i \leq \bar{\ell}_t^i \quad \forall t \\ & c_t^i, k_{t+1}^i, l_t^i, n_t^i \geq 0 \quad \forall t \\ & \hat{b}_0^i = 0, \hat{k}_0^i = \bar{k}_0^i \quad \text{given} \end{aligned}$$

- 2.) Prices are s.t. the firm chooses $\hat{Z}^F$ to minimize costs and make zero profits:

$$\begin{aligned} \hat{y}_t^f &= F(\hat{k}_t^f, \hat{n}_t^f) \quad \forall t \\ \tilde{r}_t^k &= F_k(\hat{k}_t^f, \hat{n}_t^f) \quad \forall t \\ \tilde{w}_t &= F_n(\hat{k}_t^f, \hat{n}_t^f) \quad \forall t \end{aligned}$$

- 3.) Markets clear in each period:

$$\begin{aligned} \left(\sum_{i=1}^N \hat{c}_t^i \right) + \hat{k}_{t+1}^f - (1 - \delta)\hat{k}_t^f &= F(\hat{k}_t^f, \hat{n}_t^f) = \hat{y}_t^f \quad \forall t \quad (\text{Goods Market}) \\ \sum_{i=1}^N \hat{k}_t^i &= \hat{k}_t^f \quad \forall t \quad (\text{Capital Market}) \\ \sum_{i=1}^N \hat{n}_t^i &= \hat{n}_t^f \quad \forall t \quad (\text{Labor Market}) \\ \sum_{i=1}^N \hat{b}_t^i &= 0 \quad \forall t \quad (\text{Bonds Market}) \end{aligned}$$

⁴Notice that I use a "tilde" here to help you remember that the SME wages and rental rates are not necessarily the same as those in an ADE. In fact, given the same endowments, we know the relationship between these are: $\tilde{w}_t = \hat{w}_t / \hat{p}_t$ and $\tilde{r}_t^k = \hat{r}_t^k / \hat{p}_t$.

Consider the market clearing conditions. We can rewrite the goods market clearing condition as:

$$\begin{aligned}
& \left(\sum_{i=1}^N \hat{c}_t^i \right) + \left(\sum_{i=1}^N \hat{k}_{t+1}^i \right) - (1 - \delta) \left(\sum_{i=1}^N \hat{k}_t^i \right) = F \left(\sum_{i=1}^N \hat{k}_t^i, \sum_{i=1}^N \hat{n}_t^i \right) \quad \forall t \\
& \iff \left(\sum_{i=1}^N \hat{c}_t^i \right) + \left(\sum_{i=1}^N \hat{k}_{t+1}^i - (1 - \delta) \hat{k}_t^i \right) = F \left(\sum_{i=1}^N \hat{k}_t^i, \sum_{i=1}^N \hat{n}_t^i \right) \quad \forall t \\
& \iff \sum_{i=1}^N \left(\hat{c}_t^i + \hat{k}_{t+1}^i - (1 - \delta) \hat{k}_t^i \right) = F \left(\sum_{i=1}^N \hat{k}_t^i, \sum_{i=1}^N \hat{n}_t^i \right) \quad \forall t \\
& \iff \sum_{i=1}^N \left(\hat{c}_t^i + \hat{i}_t^i \right) = F \left(\sum_{i=1}^N \hat{k}_t^i, \sum_{i=1}^N \hat{n}_t^i \right) \quad \forall t
\end{aligned}$$

Where we use the formula for household i 's gross investment in period t :

$$\hat{i}_t^i = \hat{k}_{t+1}^i - (1 - \delta) \hat{k}_t^i$$

We can clearly see that the amount produced by the firm at period t (RHS) is equal to the sum of consumption and gross investment across all of the N households at period t (LHS).

We also notice that the firm rents capital services and employs labor from each of the N households. In equilibrium, the amount of capital services supplied must equal the amount demanded, and the amount of labor supplied must equal the amount demanded.

The capital market clearing condition says that the total amount of capital services supplied by the households $\sum_{i=1}^N \hat{k}_t^i$ is equal to the amount of capital services demanded by the firm $\hat{k}_t^f$ across all periods. Thus in equilibrium, we require that:

$$\sum_{i=1}^N \hat{k}_t^i = \hat{k}_t^f \quad \forall t$$

Similarly, the labor market clearing condition says that the total amount of labor supplied by the households $\sum_{i=1}^N \hat{n}_t^i$ is equal to the amount of labor demanded by the firm $\hat{n}_t^f$ in all periods. In other words, in equilibrium:

$$\sum_{i=1}^N \hat{n}_t^i = \hat{n}_t^f \quad \forall t$$

Finally, in a SME, we need the bond market clearing condition. For all periods, we require the amount of borrowing and lending to sum to zero across all households in equilibrium:

$$\sum_{i=1}^N \hat{b}_t^i = 0 \quad \forall t$$

3 Pareto Efficiency and SPP in a Production Economy and N Infinitely Lived Agents

In the infinitely-lived model with N types of households, Pareto efficiency is defined as follows:

Definition: Households' (real)⁵ allocations $\hat{Q}^{Hi} \equiv \{\hat{c}_t^i, \hat{k}_{t+1}^i, \hat{n}_t^i, \hat{l}_t^i\}_{t=0}^\infty \forall i$ and firm production plan $\hat{Z}^F \equiv \{\hat{y}_t^f, \hat{k}_t^f, \hat{n}_t^f\}_{t=0}^\infty$ is **Pareto Efficient** if:

1.) $\{\hat{Q}^{Hi}\}_{i \in I}$ and $\hat{Z}^F$ are **feasible**:

$$\left(\sum_{i=1}^N \hat{c}_t^i \right) + \hat{k}_{t+1}^f - (1 - \delta) \hat{k}_t^f = F(\hat{k}_t^f, \hat{n}_t^f) = \hat{y}_t^f \quad \forall t$$

$$\hat{k}_0^i \leq \bar{k}_0^i \quad \forall i \in I$$

$$\sum_{i=1}^N \hat{k}_t^i = \hat{k}_t^f \quad \forall t$$

$$\sum_{i=1}^N \hat{n}_t^i = \hat{n}_t^f \quad \forall t$$

$$\hat{c}_t^i, \hat{k}_t^i, \hat{n}_t^i, \hat{l}_t^i \geq 0 \quad \forall t, \forall i \in I$$

$$\hat{n}_t^i + \hat{l}_t^i \leq \ell_t^i \quad \forall t, \forall i \in I$$

2.) There doesn't exist other feasible household (real) allocations $\tilde{Q}^{Hi} \equiv \{\tilde{c}_t^i, \tilde{k}_{t+1}^i, \tilde{n}_t^i, \tilde{l}_t^i\}_{t=0}^\infty \forall i$ and firm production plans $\tilde{Z}^F \equiv \{\tilde{y}_t^f, \tilde{k}_t^f, \tilde{n}_t^f\}_{t=0}^\infty$ such that:

$$\sum_{t=0}^\infty \beta^t u_i(\tilde{c}_t^i, \tilde{l}_t^i) \geq \sum_{t=0}^\infty \beta^t u_i(\hat{c}_t^i, \hat{l}_t^i) \quad \forall i \in I$$

$$\sum_{t=0}^\infty \beta^t u_j(\tilde{c}_t^j, \tilde{l}_t^j) > \sum_{t=0}^\infty \beta^t u_j(\hat{c}_t^j, \hat{l}_t^j) \quad \text{some } j \in I$$

When $N = 1$, this is the same Pareto efficient definition as shown by Professor Kehoe in his midterm solutions. We can now define a social planner's problem in this environment; however, this problem does not solve for the individual capital stocks $\hat{k}_t^i$; it can only solve for the aggregate capital stock $\hat{k} = \sum_{i=1}^N \hat{k}_t^i$.

⁵“Real” as in not including bonds. For Pareto efficiency, we only care about physical (“real”) constraints like the goods, capital, and labor market clearing conditions.

Definition: Given non-negative Pareto weights⁶ $(\alpha_1, \alpha_2, \dots, \alpha_N)$ and initial capital stock $\bar{k}_0^i \forall i \in I$, the **Social Planner's Problem** chooses $\hat{W}^{Hi} \equiv \{\hat{c}_t^i, \hat{n}_t^i, \hat{l}_t^i\}_{t=0}^\infty \forall i \in I$ and aggregate capital $\{\hat{k}_t\}_{t=0}^\infty$ to solve:

$$\begin{aligned} & \max_{\{W^{Hi}\}_{i \in I}, \{k_t\}_{t=0}^\infty} \sum_{i=1}^N \alpha_i \left(\sum_{t=0}^\infty \beta^t u_i(c_t^i, l_t^i) \right) \\ \text{s.t.} \quad & \left(\sum_{i=1}^N c_t^i \right) + k_{t+1} - (1 - \delta)k_t \leq F(k_t, \sum_{i=1}^N n_t^i) \\ & n_t^i + l_t^i \leq \ell_t^i \quad \forall t, \forall i \in I \\ & c_t^i, n_t^i, l_t^i, k_t \geq 0 \quad \forall t, \forall i \in I \\ & \hat{k}_0 \leq \bar{k}_0 = \sum_{i=1}^N \bar{k}_0^i \end{aligned}$$

Again, note that the social planner solves for $\{\hat{c}_t^i, \hat{n}_t^i, \hat{l}_t^i\}_{t=0}^\infty \forall i \in I$ and optimal aggregate capital. If we instead assumed that the social planner solves for $k_t^i \forall t, i$ and plugged in $k_t = \sum_{i=1}^N k_t^i \forall t$, we would find that the exact values of k_t^i are indeterminate. Therefore, using the social planner approach we can only solve for optimal *aggregate* capital.

We will now write the Bellman equation for a **stationary** dynamic programming problem.⁷ To write the Bellman equation for a stationary problem, we need ℓ_t^i to be constant across time for all households.⁸ Here is the Bellman equation under this simplifying assumption:⁹

$$\begin{aligned} V(k) = & \max_{k', \{c^i, n^i, l^i\}_{i \in I}} \left\{ \sum_{i=1}^N \alpha_i \left(\sum_{t=0}^\infty \beta^t u_i(c^i, l^i) \right) + \beta V(k') \right\} \\ \text{s.t.} \quad & \left(\sum_{i=1}^N c^i \right) + k' - (1 - \delta)k = F(k, \sum_{i=1}^N n^i) \\ & n^i + l^i \leq \ell^i \quad \forall i \in I \\ & c^i, n^i, l^i, k' \geq 0 \quad \forall i \in I \end{aligned}$$

Once we find the policy function for capital for this problem, $k'^* = g_k(k)$, we can plug in $\hat{k}_0 = \bar{k}_0 = \sum_{i=1}^N \bar{k}_0^i$ and solve for $\hat{k}_1 = g_k(\hat{k}_0)$. We can keep plugging back into the policy function to get the optimal aggregate capital from the planner's problem $\{\hat{k}_t\}_{t=0}^\infty$.

⁶It is easiest to assume that $\sum_{i=1}^N \alpha_i = 1$.

⁷We assume that the utility functions and production function meet the required conditions described in SLP.

⁸Or, if labor is inelastically supplied, we only need $\sum_{i=1}^N \ell_t^i$ to be constant $\forall t$.

⁹See Question 3(e) on the 2021 Final to see when we may need multiple value functions. In that exercise, since labor is not supplied inelastically and since time endowments vary across periods, we will need a value function for both odd and even periods.

4 ADE/SME with Production and N Types of Consumers in Two-Period OLG

In this section, we assume that there are N types of consumers in an OLG model where each consumer lives two periods (other than the initial generations). Thus there are N initial generations, and each of these initial generations are endowed with fiat money m^i . As above, we will assume that time endowments are $\ell_t^{it}, \ell_{t+1}^{it}$ for each generation.

Definition: An ADE with production consists of:

- i.) An allocation for each generation for all types: $(\hat{c}_1^{i0}, \hat{n}_1^{i0}, \hat{l}_1^{i0}, \hat{k}_1^{i0}), \{\hat{c}_t^{it}, \hat{n}_t^{it}, \hat{l}_t^{it}, \hat{k}_{t+1}^{it}, \hat{c}_{t+1}^{it}, \hat{n}_{t+1}^{it}, \hat{l}_{t+1}^{it}\}_{t=1}^{\infty} \forall i \in I$
 - ii.) Firm Allocation: $\hat{Z}^F \equiv \{\hat{y}_t^f, \hat{k}_t^f, \hat{n}_t^f\}_{t=0}^{\infty}$
 - iii.) Price System: $P \equiv \{\hat{p}_t, \hat{r}_t^k, \hat{w}_t\}_{t=0}^{\infty}$
- with $\hat{k}_1^{i0} = \bar{k}_1^{i0} \forall i$ and $m^i \forall i$ given, such that:

- 1.) Given P , initial generation of each type $i \in I$ chooses $(\hat{c}_1^{i0}, \hat{n}_1^{i0}, \hat{l}_1^{i0}, \hat{k}_1^{i0})$ to solve:

$$\begin{aligned} & \max_{c_1^{i0}, n_1^{i0}, l_1^{i0}} U(c_1^{i0}, l_1^{i0}) \\ \text{s.t. } & \hat{p}_1 c_1^{i0} \leq \hat{w}_1 n_1^{i0} + (\hat{p}_1(1 - \delta) + \hat{r}_1^k) \bar{k}_1^{i0} + m^i \\ & n_1^{i0} + l_1^{i0} \leq \ell_1^{i0} \\ & c_1^{i0}, n_1^{i0}, l_1^{i0} \geq 0 \\ & \bar{k}_1^{i0} \text{ given} \end{aligned}$$

- 2.) Given P , for each type $i \in I$ generations $t \geq 1$ choose $\hat{Z}^{ti} \equiv (\hat{c}_t^{it}, \hat{n}_t^{it}, \hat{l}_t^{it}, \hat{k}_{t+1}^{it}, \hat{c}_{t+1}^{it}, \hat{n}_{t+1}^{it}, \hat{l}_{t+1}^{it})$ to solve:

$$\begin{aligned} & \max_{Z^{it}} u(c_t^{it}, l_t^{it}) + u(c_{t+1}^{it}, l_{t+1}^{it}) \\ \text{s.t. } & \hat{p}_t(c_t^{it} + k_{t+1}^{it}) + \hat{p}_{t+1} c_{t+1}^{it} \leq \hat{w}_t n_t^{it} + \hat{w}_{t+1} n_{t+1}^{it} + (\hat{p}_{t+1}(1 - \delta) + \hat{r}_{t+1}^k) k_{t+1}^{it} \\ & n_t^{it} + l_t^{it} \leq \ell_t^{it} \\ & n_{t+1}^{it} + l_{t+1}^{it} \leq \ell_{t+1}^{it} \\ & c_t^{it}, n_t^{it}, l_t^{it}, c_{t+1}^{it}, n_{t+1}^{it}, l_{t+1}^{it}, k_{t+1}^{it} \geq 0 \end{aligned}$$

- 3.) Prices are such that the firm chooses $\hat{Z}^F$ to minimize costs and make zero profits:

$$\begin{aligned} \hat{y}_t^f &= F(\hat{k}_t^f, \hat{n}_t^f) \quad \forall t \\ \hat{r}_t^k &= \hat{p}_t F_k(\hat{k}_t^f, \hat{n}_t^f) \quad \forall t \\ \hat{w}_t &= \hat{p}_t F_n(\hat{k}_t^f, \hat{n}_t^f) \quad \forall t \end{aligned}$$

4.) Markets clear $\forall t$

$$\sum_{i=1}^N \left(\hat{c}_t^{it} + \hat{c}_t^{i,t-1} + \hat{k}_{t+1}^{it} - (1 - \delta) \hat{k}_t^{i,t-1} \right) = F(\hat{k}_t^f, \hat{n}_t^f) = \hat{y}_t^f \quad \forall t$$

$$\sum_{i=1}^N \hat{k}_t^{i,t-1} = \hat{k}_t^f \quad \forall t$$

$$\sum_{i=1}^N \left(\hat{n}_t^{it} + \hat{n}_t^{i,t-1} \right) = \hat{n}_t^f \quad \forall t$$

Now I will define a Sequential Markets Equilibrium with production in a two-period OLG setting with N types of consumers.

Definition: A SME with production consists of:

i.) An allocation for each generation for all types: $(\hat{c}_1^{i0}, \hat{n}_1^{i0}, \hat{l}_1^{i0}, \hat{k}_1^{i0})$,

$\{\hat{c}_t^{it}, \hat{n}_t^{it}, \hat{l}_t^{it}, \hat{k}_{t+1}^{it}, \hat{b}_{t+1}^{it}, \hat{c}_{t+1}^{it}, \hat{n}_{t+1}^{it}, \hat{l}_{t+1}^{it}\}_{t=1}^{\infty} \quad \forall i \in I$

ii.) Firm Allocation: $\hat{Z}^F \equiv \{\hat{y}_t^f, \hat{k}_t^f, \hat{n}_t^f\}_{t=0}^{\infty}$

iii.) Price System: $P \equiv \{\tilde{r}_t^b, \tilde{r}_t^k, \tilde{w}_t\}_{t=0}^{\infty}$

with $\hat{k}_1^{i0} = \bar{k}_1^{i0} \quad \forall i$, $\hat{b}_1^{i0} = \bar{b}_1^{i0} = 0 \quad \forall i$ and $m^i \quad \forall i$ given, such that:

1.) Given P , initial generation of each type $i \in I$ chooses $(\hat{c}_1^{i0}, \hat{n}_1^{i0}, \hat{l}_1^{i0}, \hat{k}_1^{i0})$ to solve:

$$\begin{aligned} & \max_{\hat{c}_1^{i0}, \hat{n}_1^{i0}, \hat{l}_1^{i0}} U(\hat{c}_1^{i0}, \hat{l}_1^{i0}) \\ \text{s.t. } & \hat{c}_1^{i0} \leq \tilde{w}_1 \hat{n}_1^{i0} + (1 + \tilde{r}_1^k - \delta) \bar{k}_1^{i0} + m^i \\ & \hat{n}_1^{i0} + \hat{l}_1^{i0} \leq \ell_1^{i0} \\ & \hat{c}_1^{i0}, \hat{n}_1^{i0}, \hat{l}_1^{i0} \geq 0 \\ & \bar{b}_1^0 = 0, \bar{k}_1^0 \text{ given} \end{aligned}$$

2.) Given P , for each type $i \in I$ generations $t \geq 1$ choose $\hat{Z}^{ti} \equiv (\hat{c}_t^{it}, \hat{n}_t^{it}, \hat{l}_t^{it}, \hat{k}_{t+1}^{it}, \hat{b}_{t+1}^{it}, \hat{c}_{t+1}^{it}, \hat{n}_{t+1}^{it}, \hat{l}_{t+1}^{it})$ to solve:

$$\begin{aligned} & \max_{\hat{Z}^{ti}} u(\hat{c}_t^{it}, \hat{l}_t^{it}) + u(\hat{c}_{t+1}^{it}, \hat{l}_{t+1}^{it}) \\ \text{s.t. } & \hat{c}_t^{it} + \hat{k}_{t+1}^{it} + \hat{b}_{t+1}^{it} \leq \tilde{w}_t \hat{n}_t^{it} \\ & \hat{c}_{t+1}^{it} \leq \tilde{w}_{t+1} \hat{n}_{t+1}^{it} + (1 + \tilde{r}_{t+1}^k - \delta) \hat{k}_{t+1}^{it} + (1 + \tilde{r}_{t+1}^b) \hat{b}_{t+1}^{it} \\ & \hat{n}_t^{it} + \hat{l}_t^{it} \leq \ell_t^{it} \\ & \hat{n}_{t+1}^{it} + \hat{l}_{t+1}^{it} \leq \ell_{t+1}^{it} \\ & \hat{c}_t^{it}, \hat{n}_t^{it}, \hat{l}_t^{it}, \hat{c}_{t+1}^{it}, \hat{n}_{t+1}^{it}, \hat{l}_{t+1}^{it}, \hat{k}_{t+1}^{it} \geq 0 \end{aligned}$$

3.) Prices are such that the firm chooses $\hat{Z}^F$ to minimize costs and make zero profits:

$$\begin{aligned} \hat{y}_t^f &= F(\hat{k}_t^f, \hat{n}_t^f) \quad \forall t \\ \tilde{r}_t^k &= F_k(\hat{k}_t^f, \hat{n}_t^f) \quad \forall t \\ \tilde{w}_t &= F_n(\hat{k}_t^f, \hat{n}_t^f) \quad \forall t \end{aligned}$$

4.) Market clears $\forall t$

$$\sum_{i=1}^N \left(\hat{c}_t^{it} + \hat{c}_t^{i,t-1} + \hat{k}_{t+1}^{it} - (1 - \delta) \hat{k}_t^{i,t-1} \right) = F(\hat{k}_t^f, \hat{n}_t^f) = \hat{y}_t^f \quad \forall t$$

$$\sum_{i=1}^N \hat{k}_t^{i,t-1} = \hat{k}_t^f \quad \forall t$$

$$\sum_{i=1}^N \left(\hat{n}_t^{it} + \hat{n}_t^{i,t-1} \right) = \hat{n}_t^f \quad \forall t$$

$$\sum_{i=1}^N \hat{b}_2^{i1} = \sum_{i=1}^N m^i$$

$$\sum_{i=1}^N \hat{b}_{t+1}^{it} = \left[\prod_{\tau=2}^t (1 + \hat{r}_\tau^b) \right] \left(\sum_{i=1}^N m^i \right) \quad \forall t \geq 2$$

This completes the SME definition. Notice that the bond market clearing condition implies that for $N \geq 2$, trade can still occur between households even when there is no fiat money ($m^i = 0 \quad \forall i$).