

Why are we doing this?

Zach Swartz
Econ 8105
Adapted from
Amol Amol's &
Alex Wudiz's notes.

- Infinitely lived with production:

We started with two types of CE and showed that they were equivalent given same endowments:

$$(ADE \Leftrightarrow SME)$$

Then we showed that these equilibria could be mapped to a SPP by the welfare theorems:

$\stackrel{SWT}{\Leftarrow}$

$\uparrow$
social planner's problem

$\Rightarrow$ SPP

$\stackrel{FWT}{\Rightarrow}$

Now, we want to map these SPP to a dynamic programming problem:

$$\stackrel{\text{check SLP}}{\Leftrightarrow} DP$$

Just as the welfare theorems only hold under certain assumptions, specific conditions have to be met in order for $SPP \Leftrightarrow DP$ to be true. SLP discusses these conditions.

So we have:

$$(ADE \Leftrightarrow SME) \stackrel{\stackrel{SWT}{\Leftarrow}}{\stackrel{FWT}{\Rightarrow}} SPP \stackrel{\stackrel{SWT}{\Leftarrow}}{\stackrel{SLP}{\Rightarrow}} DP. \text{ Let's use this idea:}$$

PS 3 Q2 : We want to calculate a SME with production.

We map $SME \Rightarrow SPP \Rightarrow DP$

Then we find the DP policy functions and use $DP \Rightarrow SPP \Rightarrow SME$.

↳ Solved $g(k_t) = k_{t+1}^* = \alpha \beta \theta k_t^\alpha = \alpha \beta y_t^f$ using DP problem.

$$c(k_t) = c_t = \theta k_t^\alpha - k_{t+1}^* = \theta k_t^\alpha - \alpha \beta \theta k_t^\alpha \\ = \theta k_t^\alpha (1 - \alpha \beta)$$

$$\Rightarrow c(k_t) = (1 - \alpha \beta) \theta k_t^\alpha = (1 - \alpha \beta) y_t^f$$

Household eats and invests a constant share of total output.

Using the initial capital stock, we can solve $g(k_t)$ forward:

$$k_t^* = (\alpha \beta \theta)^{\frac{1-\alpha^*}{1-\alpha}} k_0^{\alpha^*} \quad \forall t \geq 1$$

SME: $\hat{C}_t = c(k_t)$

$$\hat{k}_{t+1} = g(k_t)$$

$$\hat{b}_t = 0$$

$$\hat{r}_t^k = \alpha \theta g(k_{t-1})^{\alpha-1}$$

$$\hat{w}_t = (1 - \alpha) \theta g(k_{t-1})^\alpha$$

$$1 + \hat{r}_t^b = 1 + \hat{r}_t^k - \delta = \hat{r}_t^k \quad \text{by no arbitrage}$$

$$\Rightarrow \hat{r}_t^b = \hat{r}_t^k - 1$$

Using our ADE/SME equivalence results, we can calculate the ADE:

$$\underline{ADE} : \tilde{c}_t = c(k_t)$$

$$\tilde{k}_{t+1} = g(k_t)$$

$$\tilde{p}_0 = 1$$

$$\tilde{p}_t = \frac{1}{\prod_{s=1}^t (1 + \tilde{r}_s^k)} \quad \forall t \geq 1$$

$$\tilde{r}_t^k = \tilde{p}_t \alpha \theta g(k_{t-1})^{\alpha-1}$$

$$\tilde{w}_t = \tilde{p}_t (1 - \alpha) \theta g(k_{t-1})^{\alpha}$$

We see that solving a dynamic programming problem allowed us to calculate competitive equilibria because the mapping $CE \Leftrightarrow SPP \Leftrightarrow DP$ holds in this case.

Steady States

A steady state is an equilibrium such that optimal quantities are constant over time.

In an SME w/ 1 infinitely lived agent and inelastic labor supply, this means:

$$\left. \begin{aligned} \hat{C}_t &= \hat{C}_{t+1} = \hat{C} & \forall t \\ \hat{k}_t &= \hat{k}_{t+1} = \hat{k} & \forall t \\ \hat{b}_t &= \hat{b}_{t+1} = \hat{b} & \forall t \end{aligned} \right\} HH$$

$$\left. \begin{aligned} k_t^f &= k_{t+1}^f = \hat{k}^f & \forall t \\ \hat{y}_t^f &= \hat{y}_{t+1}^f = \hat{y}^f & \forall t \end{aligned} \right\} \text{Firm}$$

In this equilibrium, since prices are functions of constant variables, they are also constant.

As I mentioned in class, depending on the market structure, prices do not necessarily have to be constant in a steady state. For example, in an ADE, we require $\hat{p}_t \rightarrow 0$ as $t \rightarrow \infty$ quick enough so that the RHS & LHS of the ADE budget constraint converges.

$$\left. \begin{aligned} \hat{r}_t^k &= \hat{r}_{t+1}^k = \hat{r}^k & \forall t \\ \hat{w}_t &= \hat{w}_{t+1} = \hat{w} & \forall t \\ \hat{r}_t^b &= \hat{r}_{t+1}^b = \hat{r}^b & \forall t \end{aligned} \right\} \text{Price System}$$

Use FOCs and MC to solve for SS:

Euler equations: $\frac{\hat{c}_{t+1}}{\beta \hat{c}_t} = 1 + \hat{r}_{t+1}^b = \hat{r}_{t+1}^k$

$$\Rightarrow \frac{1}{\beta} = 1 + \hat{r}^b = \hat{r}^k ;$$

$$\hat{w}_t = (1-\alpha)\theta \hat{k}_t^\alpha \Rightarrow \hat{w} = (1-\alpha)\theta \hat{k}^\alpha$$

$$\hat{r}_t^k = \alpha\theta \hat{k}_t^{\alpha-1} \Rightarrow \frac{1}{\beta} = \alpha\theta \hat{k}^{\alpha-1}$$

$$\Rightarrow \hat{k} = \left(\frac{1}{\beta\alpha\theta}\right)^{\frac{1}{\alpha-1}} = \hat{k}^f ; \hat{y}^f = \theta \hat{k}^\alpha$$

$$\Rightarrow \hat{c} = \theta \hat{k}^\alpha - \hat{k} ; \hat{b} = 0$$

To summarize, the SME steady state satisfies $\forall t$:

$$\left. \begin{aligned} \hat{c}_t = \hat{c} &= \theta \hat{k}^\alpha - \hat{k} \\ \hat{b}_t = \hat{b} &= 0 \\ \hat{k}_t = \hat{k} &= \left(\frac{1}{\beta\alpha\theta}\right)^{\frac{1}{\alpha-1}} = (\alpha\theta\beta)^{\frac{1}{1-\alpha}} \end{aligned} \right\} \text{Household}$$

$$\left. \begin{aligned} \hat{k}_t^f = \hat{k}^f &= \hat{k} \\ \hat{y}_t^f = \hat{y}^f &= \theta \hat{k}^\alpha \end{aligned} \right\} \text{Firm}$$

$$\left. \begin{aligned} \hat{r}_t^b = \hat{r}^b &= 1/\beta \\ \hat{r}_t^k = \hat{r}^k &= 1/\beta - 1 \\ \hat{w}_t = \hat{w} &= (1-\alpha)\theta \hat{k}^\alpha \end{aligned} \right\} \text{Price system}$$

Thus we infer that if the initial (given) quantities are equal to their steady state values, which in this case means $\bar{b}_0 = 0$ and $\bar{k}_0 = (\alpha\beta\theta)^{\frac{1}{1-\alpha}}$, then the economy will optimally remain at those values for all periods.

We know from above that optimal capital stock at period t (given $\bar{k}_0$) satisfies:

$$k_t^* = (\alpha\beta\theta)^{\frac{1-\alpha^t}{1-\alpha}} \bar{k}_0^{\alpha^t} \quad \forall t \geq 1$$

Now, suppose the initial capital stock was equal to the steady state value, $\bar{k}_0 = (\alpha\beta\theta)^{\frac{1}{1-\alpha}}$.

We infer that we should get $\hat{k}_t = (\alpha\beta\theta)^{\frac{1}{1-\alpha}} \quad \forall t$, but let's test this using the policy function:

$$\begin{aligned} k_t^* &= (\alpha\beta\theta)^{\frac{1-\alpha^t}{1-\alpha}} \bar{k}_0^{\alpha^t} \quad \forall t \geq 1 \\ &= (\alpha\beta\theta)^{\frac{1-\alpha^t}{1-\alpha}} \left((\alpha\beta\theta)^{\frac{1}{1-\alpha}} \right)^{\alpha^t} = (\alpha\beta\theta)^{\frac{1-\alpha^t + \alpha^t}{1-\alpha}} = (\alpha\beta\theta)^{\frac{1}{1-\alpha}} \end{aligned}$$

Thus, given $\bar{k}_0 = (\alpha\beta\theta)^{\frac{1}{1-\alpha}}$, optimal capital is equal to

$$\hat{k}_t = (\alpha\beta\theta)^{\frac{1}{1-\alpha}} \quad \text{for all } t, \text{ just as we expected.}$$

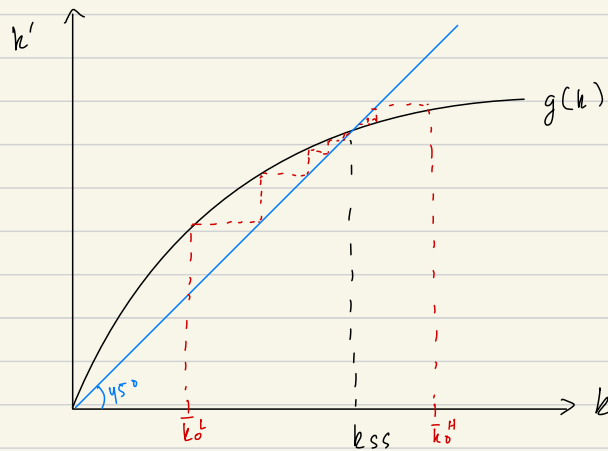
As a look-ahead to 8106, we can also see that for any initial $\bar{k}_0$, $\hat{k}_t$ and $\hat{c}_t$ will converge to k_{ss} and c_{ss} as $t \rightarrow \infty$.

To see this, consider the capital policy function:

$$k_{t+1}^* = g(k_t) = \alpha \beta \theta k_t^\alpha.$$

Since $\alpha, \beta, \theta > 0$ and $\alpha \in (0, 1)$ by assumption,

the graph of $k' = g(k)$ is:



We see that if we start at $k_0^L < k_{ss}$ or $k_0^H > k_{ss}$, the capital stock converges to k_{ss} .

And since $k_t^* \rightarrow k_{ss}$ as $t \rightarrow \infty$, then $c_t^* \rightarrow c_{ss}$ as well, since c_t is a function of k_t .

We can also clearly see that k_{ss} is the fixed point $g(k_{ss}) = k_{ss}$.

Dynamic Programming Part II

Blackwell's sufficient conditions:

This is defined for our simple examples. See SLP for general definition.

Let $K \subseteq \mathbb{R}_+$, and let $C(K)$ be a space of bounded functions $v: K \rightarrow \mathbb{R}$, with the sup norm.

Let $T: C(K) \rightarrow C(K)$ be an operator satisfying:

i) Monotonicity: If $v, w \in C(K)$ satisfy $v(k) \leq w(k) \quad \forall k \in K$, then: $(Tv)(k) \leq (Tw)(k) \quad \forall k \in K$

ii) Discounting: $\exists$ some $\beta \in (0, 1)$ such that $[T(v+a)](k) \leq (Tv)(k) + \beta a \quad \forall v \in C(K), a \geq 0, k \in K$

Then T is a contraction with modulus β . ■

[Note: $(v+a)(k) = v(k) + a$]

Write our operator from before:

[E.g.: $f(k) = \theta k^\alpha + (1-\theta)k$]

$$\begin{aligned} T(W)(k) &= \max \{ u(f(k) - k') + \beta W(k') \} \\ \text{s.t. } k' &\in \{ k' \in K \mid 0 \leq k' \leq f(k) \} \\ &= u(f(k) - g(k)) + \beta W(g(k)) ; k^* = g(k) \end{aligned}$$

We need this to be bounded $\forall W(k) \in C(K)$, $V: K \rightarrow \mathbb{R}_+$ where $C(K)$ is space of bounded functions.

Since $W(k) \in C(K)$, by definition $W(k)$ is bounded. We just need to ensure $u(f(k_t) - k')$ is bounded at the optimum. We assume u is continuous, st. increasing, is unbounded from above & $u(0)$ is finite. E.g. $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$, $\sigma \in (0, 1)$

Also assume u does not depend on leisure, so labor supplied inelastically.

Thus we set $F(k_t, l_t) = \theta k_t^\alpha l_t^{1-\alpha}$ to $F(k_t, 1) = \theta k_t^\alpha$; $l_t = 1$.

We will ultimately bound $u(\cdot)$ by setting a finite upper bound for $K \subseteq \mathbb{R}_+$, the set of possible values of capital.

Maximum Sustainable Capital

How do we pick this bound?

Both finite

We already know that we should set $K = [0, \max\{k_0, \bar{k}\}]$, where $\bar{k}$ is the maximum sustainable capital.

However, a good question was asked in recitation -

why don't we just pick an arbitrary upper bound, say $\bar{k}$, and set $K = [0, \bar{k}]$?

One reason is that we do not want the agent's optimal choice to be impacted by our choice of $\bar{k}$.

If $\bar{k}$ is too low, then the agent might get "stuck" at this value of capital, even though they want to have a higher capital stock. To see this, remember that the k' choice given k must satisfy $k' \in \{k' \in K \mid 0 \leq k' \leq f(k)\}$.

If $K = [0, \bar{k}]$ and $\bar{k}$ is too low, the agent may want to choose $k'' > \bar{k}$; but it is not an option since k'' is not in K .

To summarize, we want to pick the upper bound of the K set so that it doesn't affect the agent's choices.

This is why we are interested in the maximum sustainable capital (MSC). It gives us an explicit value for the upper bound on K that won't affect the agent's decisions.

At the maximum sustainable capital, none of the output is spent on consumption; $C_t = 0$.

All of output is spent on investment.

From feasibility, this means:

$$\cancel{C_t} + k_{t+1} = f(k_t) = \theta k_t^\alpha + (1-\delta)k_t$$

$$\Rightarrow k_{t+1} = \theta k_t^\alpha + (1-\delta)k_t$$

$$\Rightarrow k_{t+1} - (1-\delta)k_t = i_t = \theta k_t^\alpha$$

Since i_t is gross investment, we see that all of output is spent re-investing in the capital stock.

But we also need the maximum sustainable capital to meet another condition:

After investing all of output back into capital, the capital stock is unchanged.

At the maximum sustainable capital value, even if the agent invests all output into capital, the capital stock remains the same!

This is why it is the maximum sustainable capital; it is the highest level of capital stock that can be sustained over time.

Additionally, we want to make sure that if the initial capital stock $\bar{k}_0$ is higher/lower than $\tilde{k}$, and if the agent only spends output on investment $\forall t$, then $\bar{k}_0, k_1 = \theta k_0^a + (1-\delta)k_0, k_2 = \theta k_1^a + (1-\delta)k_1, k_3, \dots, k_n, \dots$

converges to $\tilde{k}$. For example, we don't want to have

$k_1 = f(k_0), k_2 = f(k_1), \dots, k_t = f(k_{t-1}), \dots$ converging to $+\infty$ if $\bar{k}_0 > \tilde{k}$.

If $\bar{k}_0 > \tilde{k}$ and $k_{t+1} = f(k_t) \rightarrow +\infty$, then the agent may want to pick k_t such that $k_t \rightarrow \infty$; but this is not possible since K is bounded above by $\max\{\bar{k}_0, \tilde{k}\}$. Again, we want the upper bound of K so that it never affects the agent's decision making.

Now we are ready to define the MSC:

• The maximum sustainable capital $\tilde{k}$ satisfies:

a) $\tilde{k} > 0$ (and finite)

b) $\tilde{k} = f(\tilde{k})$ ($= \theta \tilde{k}^\alpha + (1-\delta)\tilde{k}$ in our example)

c) $\forall \bar{k}_0 > 0, \lim_{t \rightarrow \infty} f(k_t) = \lim_{t \rightarrow \infty} k_{t+1} = \tilde{k}$

where $k_{t+1} = f(k_t) \quad \forall t \geq 0$

Notes on this:

a) $\tilde{k} = 0$ would not make much sense.

b) $\tilde{k} = f(\tilde{k})$ says that all of output is spent on investing in the capital stock ($c=0$).

c) Says that for any initial capital stock, if the agent only spent output on capital $\forall t$, then k_{t+1} would converge to $\tilde{k}$.

We can see (b), $k' = \tilde{k} = f(\tilde{k}) = \theta \tilde{k}^\alpha + (1-\delta)\tilde{k}$, looks a lot like a fixed point of $k' = h(k)$.

Define $k' = h(k) = \theta k^\alpha + (1-\delta)k$.

$h(k)$ is equal to next periods capital stock when $C=0$.

A non-zero fixed point for $h(k)$ satisfies:

$$h(\tilde{k}) = \tilde{k} > 0$$

$$\Leftrightarrow \theta \tilde{k}^\alpha + (1-\delta)\tilde{k} = \tilde{k} \Rightarrow \tilde{k} = \left(\frac{\theta}{\delta}\right)^{\frac{1}{\alpha-1}}$$

We can solve for this fixed point algebraically, but we need to make sure this $\tilde{k} > 0$ actually exists and satisfies the convergence property.

$$\text{Set } LHS = L(k) = \theta k^\alpha + (1-\delta)k$$

$$RHS = R(k) = k$$

Let's try to see if $\tilde{k}$ exists under certain parameters:

i) Assume $\theta > 0$, $\alpha \in (0, 1)$, $\delta \in (0, 1]$.

$$\text{We know } \lim_{k \rightarrow \infty} L'(k) = \lim_{k \rightarrow \infty} \alpha \theta k^{\alpha-1} + (1-\delta) = 1-\delta < 1$$

$$\text{and } \lim_{k \rightarrow 0} L'(k) = +\infty. \text{ We can also see}$$

that $L(k)$ is strictly increasing, concave, and continuous.

From these, we can infer that $L(k)$ crosses

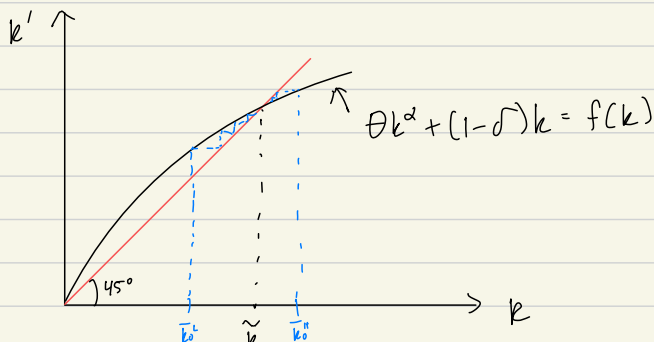
the 45° line once for $k > 0$; this is the same as

$$\exists \tilde{k} \text{ s.t. } L(\tilde{k}) = R(\tilde{k}) \Leftrightarrow \theta \tilde{k}^\alpha + (1-\delta)\tilde{k} = \tilde{k}.$$

$$\text{We find that the fixed point exists + } \tilde{k} = \left(\frac{\theta}{\delta}\right)^{\frac{1}{\alpha-1}}.$$

We still need to check the convergence property.

Using the plot below, we can see graphically that $\tilde{k}$ exists, that the convergence property holds for both $\bar{k}^L < \tilde{k}$ and $\bar{k}^H > \tilde{k}$.



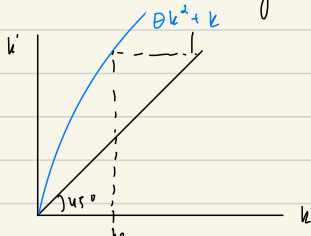
So for $\theta > 0$, $\alpha \in (0,1)$ and $\delta \in (0,1]$, MSC $\tilde{k}$ exists.

ii) What if $\alpha \in (0,1)$, $\theta > 0$, $\delta = 0$?

Does a maximum sustainable capital exist?

add) Does $\exists \tilde{k} > 0$ s.t. $\theta \tilde{k}^\alpha + \tilde{k} = \tilde{k}$?

Rearranging: $\theta \tilde{k}^\alpha = 0 \Rightarrow$ Only true when $\tilde{k} = 0$.

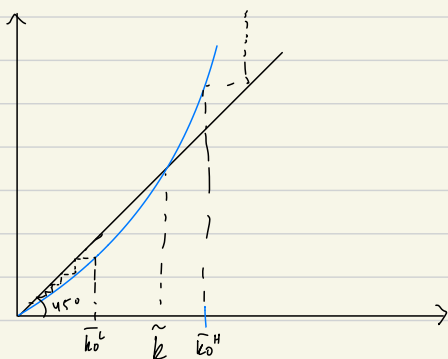


$L'(k) = \alpha \theta k^{\alpha-1} + 1 > 1 \quad \forall k \Rightarrow L(k)$ never intersects 45° for $k > 0$.

$\Rightarrow$ Thus when $\delta = 0$ a MSC does not exist.

ii) What if $\alpha > 1$, $\theta > 0$, $\delta \in (0, 1]$?

a+b) Does $\exists \tilde{k} > 0$ s.t. $\theta \tilde{k}^\alpha + (1-\delta)\tilde{k} = \tilde{k}$? $\Rightarrow$ Yes



c) Is convergence property satisfied?

$\Rightarrow$ No. If $\bar{k}_0^H > \tilde{k}$, then $k_{t+1} = f(k_t) \rightarrow +\infty$

If $\bar{k}_0^L < \tilde{k}$, then $k_{t+1} = f(k_t) \rightarrow 0$.

$\Rightarrow$ When $\alpha > 1$, MSC does not exist.

We see that for $f(k) = \theta k^\alpha + (1-\delta)k$, we must assume $\theta > 0$, $\alpha \in (0, 1)$, and $\delta \in (0, 1]$ for a maximum sustainable capital $\tilde{k}$ to exist.

Thus, we define the functions in $C(K)$ s.t.

$V: K \rightarrow \mathbb{R}_+$ where $K = [0, \max\{k_0, \tilde{k}\}]$

This implies that the highest possible value of $u(f(k) - k')$ is $\max\{u(k_0), u(\tilde{k})\}$, which is finite.

Thus we know our operator maps from a set of bounded functions to bounded functions.

To summarize, for a maximum sustainable capital to exist, ^(*) the following conditions are sufficient:

- $\delta \in (0, 1]$
- Production function $F(k)$ satisfies:

- i) Continuous, increasing, and concave in k
- ii) $F(0) = 0$ (we just always assume this)
- iii) $\lim_{k \rightarrow \infty} F_k(k) = 0$
- iv) $\lim_{k \rightarrow 0} F_k(k) = +\infty$

Thus, if we assume the above, we have implicitly bounded the utility function from above.

You can see that in our example above,

$F(k_t) = \theta k_t^\alpha$ satisfies these properties, and

we also assumed $\delta \in (0, 1]$.

^(*) For this specific case when labor is supplied inelastically.

Now that we know $T: C(K) \rightarrow C(K)$ for sure, we can apply Blackwell's sufficient conditions for a contraction

We have our operator:

$$T(W)(k) = \max_{k' \in [0, f(k)]} \{u(f(k) - k') + \beta V(k')\}$$

i) Monotonicity: Take $w, v \in C(K)$ s.t. $w(k) \leq v(k) \forall k \in K$;

$$\Rightarrow u(f(k) - k') + \beta w(k') \leq u(f(k) - k') + \beta v(k') \quad \forall k' \in [0, f(k)]$$

$$\Rightarrow \max_{0 \leq k' \leq f(k)} u(f(k) - k') + \beta w(k') \leq \max_{0 \leq k' \leq f(k)} u(f(k) - k') + \beta v(k')$$

$$\Rightarrow T(w)(k) \leq T(v)(k) \quad \forall k \in K$$



ii) Discounting: Pick $\alpha \geq 0$.

$$T(V + \alpha)(k) =$$

$$\max_{0 \leq k' \leq f(k)} \{u(f(k) - k') + \beta(V(k') + \alpha)\}$$

$$= \max_{0 \leq k' \leq f(k)} \{u(f(k) - k') + \beta V(k')\} + \beta \alpha$$

$$= T(V)(k) + \beta \alpha$$

Thus $T(V + \alpha)(k) \leq T(V)(k) + \beta \alpha$ and $\beta \in (0, 1)$ by assumption.



Thus our operator T is a contraction w/ mod β & we can apply the Contraction Mapping Theorem:

Definition: CMT: If $(C(K), \rho)$ is a complete metric space and $T: C(K) \rightarrow C(K)$ is a contraction with modulus β , then:

i) T has exactly one fixed point v in S

ii) For any initial $v_0 \in S$, $\rho(T^n(v_0), v) \leq \beta^n(v_0, v)$ ■

We can pick any initial bounded function $v_0(k)$

and we know $\lim_{n \rightarrow \infty} V_{n+1}(k) = V^*(k) = T(V^*)(k)$ s.t.

$$T(V^*)(k) = V^*(k) = \max_{0 \leq k' \leq f(k)} \{ u(f(k) - k') + \beta V^*(k') \}.$$

When we do value function iteration (VFI) on the computer, we do not perform an infinite amount of iterations. We iterate until:

$$\sup_{k \in K} |V_{n+1}(k) - V_n(k)| < \epsilon, \text{ where } \epsilon \text{ is}$$

our "tolerance." We usually choose ϵ very small, like 10^{-8} .

We know that with every iteration, $V_n(k)$ gets closer to $V^*(k)$ by the CMT.

So we choose a small ϵ , as it tells us that $V_{n+1}(k)$ and $V_n(k)$ are very, very similar.

Since we get closer to V^* each iteration, if $p(V_{n+1}(k), V_n(k))$ is very small, then $p(V_{n+2}(k), V_{n+1}(k))$ will likely also be very small. This gives us a reason to stop the iterations once $p(V_{n+1}(k), V_n(k))$ is small enough ($< \epsilon$).

See my website for Python code to help with VFI.

MatLab code is also on Alex's Wordinger's website.

↳ Note: MatLab code is more like Excel VFI and thus might be more intuitive.

Guess and Verify

In certain cases, we can "guess" the functional form of $V(k)$, and then "verify" that that guess is correct. We verify by checking that the Euler equation and TVC are satisfied.

You have already done this in PS 3.

You guessed $V(k) = a_0 + a_1 \log(k)$ and solved for a_0 and a_1 .

Next, we will do the same but for a more complicated example.

Final 2020 Q3

$$\text{SPP: } \max \sum_{t=0}^{\infty} \beta^t [\log c_t + \gamma \log(1-l_t)]$$

$$\text{s.t. } c_t + k_{t+1} \leq \theta k_t^\alpha l_t^{1-\alpha} \quad (\lambda_t)$$

$$c_t, k_{t+1} \geq 0, \quad l_t \in [0, 1] \\ k_0 \leq \bar{k}_0$$

a) Write down Euler + TVC:

FOCs:

$$[c_t]: \frac{\beta^t}{c_t} = \lambda_t$$

$$[l_t]: \beta^t \frac{\gamma}{1-l_t} = \lambda_t (1-\alpha) \theta k_t^\alpha l_t^{-\alpha}$$

$$[k_{t+1}]: \lambda_t = \lambda_{t+1} \alpha \theta k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha}$$

$$\text{Euler: } \frac{c_{t+1}}{c_t} = \beta \alpha \theta k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha}$$

$$\text{Intertemporal MRS: } \frac{c_t \gamma}{1-l_t} = (1-\alpha) \theta k_t^\alpha l_t^{-\alpha} = F_l(k_t, l_t)$$

$$\text{TVC: } \lim_{t \rightarrow \infty} \lambda_t k_{t+1} = 0$$

b) Formulate Bellman: Guess $V(k) = a_0 + a_1 \log k$.

Bellman Equation:

$$V(k) = \max_{c, l, k'} \left\{ [\ln c + \gamma \ln(1-l)] + \beta V(k') \right\}$$

$$c + k' \leq \theta k^\alpha l^{1-\alpha}$$

$$c, k' \geq 0, \quad l \in [0, 1]$$

$$V(k) = \max_{l, k'} \left\{ [\ln(\theta k^\alpha l^{1-\alpha} - k') + \gamma \ln(1-l)] + \beta V(k') \right\}$$

$$k' \in [0, \theta k^\alpha l^{1-\alpha}], \quad l \in [0, 1]$$

Guess :

$$V(k) = \max_{l, k'} \left\{ [\ln(\theta k^\alpha l^{1-\alpha} - k') + \gamma \ln(1-l)] + \beta (a_0 + a_1 \log k') \right\}$$

$$k' \in [0, \theta k^\alpha l^{1-\alpha}], \quad l \in [0, 1]$$

FOCs (all interior)

$$[k'] : \frac{1}{\theta k^\alpha l^{1-\alpha} - k'} = \frac{\beta a_1}{k'} \quad (i)$$

$$[l] : \frac{(1-\alpha)\theta k^\alpha l^{-\alpha}}{\theta k^\alpha l^{1-\alpha} - k'} = \frac{\gamma}{1-l} \quad (ii)$$

$$(i): k' = \beta a_1 \theta k^\alpha l^{1-\alpha} - \beta a_1 k' \Rightarrow k'(1 + \beta a_1) = \beta a_1 \theta k^\alpha l^{1-\alpha}$$

$$\Rightarrow k' = \frac{\beta a_1 \theta k^\alpha l^{1-\alpha}}{1 + \beta a_1} \quad (iii)$$

$$(ii): (1-\alpha)\theta k^\alpha l^{1-\alpha} - (1-\alpha)\theta k'^\alpha l^{1-\alpha} = \gamma \theta k^\alpha l^{1-\alpha} - \gamma k' \quad \text{--- (iii)}$$

$$\frac{1}{\theta k^\alpha l^{1-\alpha}} \left[(1-\alpha)\theta k^\alpha l^{1-\alpha} - (1-\alpha)\theta k'^\alpha l^{1-\alpha} \right] = \left[\gamma \theta k^\alpha l^{1-\alpha} - \frac{\gamma \beta_{a_1} \theta k^\alpha l^{1-\alpha}}{1+\beta_{a_1}} \right]$$

$$\Rightarrow \frac{(1-\alpha)}{l} - (1-\alpha) = \gamma \left(1 - \frac{\beta_{a_1}}{1+\beta_{a_1}} \right)$$

$$\Rightarrow \frac{1}{l} = \frac{\gamma}{1-\alpha} \left(\frac{1+\beta_{a_1}}{1+\beta_{a_1}} - \frac{\beta_{a_1}}{1+\beta_{a_1}} \right) + 1$$

$$\Rightarrow 1 = l \left(\frac{\gamma}{1-\alpha} \left(\frac{1}{1+\beta_{a_1}} \right) + \frac{(1-\alpha)(1+\beta_{a_1})}{(1-\alpha)(1+\beta_{a_1})} \right)$$

$$\Rightarrow l = \frac{(1-\alpha)(1+\beta_{a_1})}{\gamma + (1-\alpha)(1+\beta_{a_1})}$$

$$\text{and } k' = \frac{\beta_{a_1} \theta k^\alpha l^{1-\alpha}}{1+\beta_{a_1}}$$

Plug l and k' back into value function :

$$V(k) = \ln \left(\theta k^\alpha l^{1-\alpha} - \frac{\beta_{a_1} \theta k^\alpha l^{1-\alpha}}{1+\beta_{a_1}} \right) + \gamma \ln(l^\alpha)$$

$$+ \beta_{a_0} + \beta_{a_1} \log \left(\frac{\beta_{a_1} \theta k^\alpha l^{1-\alpha}}{1+\beta_{a_1}} \right)$$

$$= \ln \left(\theta k^\alpha l^{1-\alpha} \left(1 - \frac{\beta_{a_1}}{1+\beta_{a_1}} \right) \right) + \gamma \ln(l^\alpha)$$

$$+ \beta_{a_0} + \beta_{a_1} \log \left(\theta k^\alpha l^{1-\alpha} \left(\frac{\beta_{a_1}}{1+\beta_{a_1}} \right) \right)$$

$$\begin{aligned}
&= \ln(\theta k^\alpha l^{*\alpha(1-\alpha)}) + \ln\left(\frac{1}{1+\beta a_1}\right) + \gamma \ln(l^*) \\
&\quad + \beta a_0 + \beta a_1 \log(\theta k^\alpha l^{*\alpha(1-\alpha)}) + \beta a_1 \log\left(\frac{\beta a_1}{1+\beta a_1}\right) \\
&= \ln\left(\frac{1}{1+\beta a_1}\right) + \beta a_1 \ln\left(\frac{\beta a_1}{1+\beta a_1}\right) + \gamma \ln(l^*) + \beta a_0 \\
&\quad + \ln(\theta l^{*\alpha(1-\alpha)}) + \beta a_1 \ln(\theta l^{*\alpha(1-\alpha)}) \\
&\quad + \alpha \ln(k) + \beta a_1 \alpha \ln(k) \} \underbrace{\alpha \theta (1+\alpha\beta) \ln(k)}_{\alpha_1} \\
&= a_1 + \underbrace{a_1 \ln(k)}_{\alpha_1}
\end{aligned}$$

$$\Rightarrow a_1 = \alpha + \alpha \beta a_1$$

$$\Rightarrow a_1(1 - \alpha\beta) = \alpha$$

$$\Rightarrow a_1 = \frac{\alpha}{1 - \alpha\beta}$$

Can now solve for a_0 as well. (Try it as practice)

Now that we have a_1 , we can solve for our policy functions.

$$\begin{aligned}
l(k) = l^* &= \frac{(1-\alpha)(1+\beta a_1)}{\gamma + (1-\alpha)(1+\beta a_1)} = \frac{(1-\alpha)/(1-\alpha\beta)}{\gamma + (1-\alpha)/(1-\alpha\beta)} \\
&= \frac{(1-\alpha)}{\gamma(1-\alpha\beta) + (1-\alpha)} \in (0,1)
\end{aligned}$$

$$\begin{aligned}
 g(k) = k' &= \frac{\beta a_1 \theta k^\alpha l^{1-\alpha}}{1 + \beta a_1} = \frac{\beta \left(\frac{\alpha}{1-\alpha\beta} \right) \theta k^\alpha l^{1-\alpha}}{1 + \beta \left(\frac{\alpha}{1-\alpha\beta} \right)} \\
 &= \frac{\beta \alpha \theta k^\alpha l^{1-\alpha} / (1 + \alpha\beta)}{1 / (1 - \alpha\beta)} \\
 &= \beta \alpha \theta k^\alpha l^{1-\alpha}
 \end{aligned}$$

$$\Rightarrow g(k) = k'^* = \beta \alpha \theta k^\alpha l^{1-\alpha}$$

$$\Rightarrow c(k) = c^* = \theta k^\alpha l^{1-\alpha} - g(k) = \theta k^\alpha l^{1-\alpha} (1 - \alpha\beta)$$

$$\Rightarrow l(k) = l^* = \frac{(1-\alpha)}{r(1-\alpha\beta) + (1-\alpha)}$$

c) Verify:

$$\text{Euler: } \frac{c_{t+1}}{c_t} = \beta \alpha \theta k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha} \Leftrightarrow \frac{c_{t+1}}{c_t \beta \alpha k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha}} = 1$$

$$\text{Intertemporal MRS: } \frac{c_t \gamma}{1 - l_t} = (1-\alpha) \theta k_t^\alpha l_t^{-\alpha}$$

$$\Leftrightarrow \frac{c_t \gamma}{(1-l_t)(1-\alpha) \theta k_t^\alpha l_t^{-\alpha}} = 1$$

$$\text{TVC: } \lim_{t \rightarrow \infty} \lambda_t k_{t+1} = 0$$

Plug in the policy functions:

$$\begin{aligned}
 \text{Euler: } \frac{c_{t+1}^\alpha}{c_t^\alpha \beta \alpha k_{t+1}^{\alpha-1} l_t^{\alpha-1}} &= \frac{\cancel{\theta k_{t+1}^\alpha} \cancel{l_{t+1}^{1-\alpha}} - \beta \alpha \cancel{\theta k_{t+1}^\alpha} \cancel{l_{t+1}^{1-\alpha}}}{(\theta k_t^\alpha l_t^{1-\alpha} - \beta \alpha \theta k_t^\alpha l_t^{1-\alpha}) \beta \alpha \cancel{k_{t+1}^\alpha} \cancel{l_{t+1}^{1-\alpha}}} \\
 &= \frac{\cancel{1 - \beta \alpha}}{\beta \alpha \theta k_t^\alpha l_t^{1-\alpha} (1 - \beta \alpha) k_t^{-1}} \\
 &= \frac{k_{t+1}}{\beta \alpha \theta k_t^\alpha l_t^{1-\alpha}} = \frac{k_{t+1}}{k_{t+1}} = 1 \quad \checkmark
 \end{aligned}$$

Intratemporal MRS:

$$\begin{aligned}
 \frac{c_t^\gamma}{(1-l_t)(1-\alpha)\theta k_t^\alpha l_t^{1-\alpha}} &= \frac{\gamma(1-\alpha\beta)\cancel{\theta k_t^\alpha} \cancel{l_t^{1-\alpha}}}{(1-l_t)(1-\alpha)\cancel{\theta k_t^\alpha} \cancel{l_t^{1-\alpha}}} \\
 &= \frac{\gamma(1-\alpha\beta) l_t}{(1-\alpha)(1-l_t)} \\
 &= \gamma(1-\alpha\beta) \left(\frac{(1-\alpha)}{\gamma(1-\alpha\beta) + (1-\alpha)} \right) \\
 &\quad \frac{(1-\alpha) \left(1 - \frac{(1-\alpha)}{\gamma(1-\alpha\beta) + (1-\alpha)} \right)}{\gamma(1-\alpha\beta)} \\
 &= 1 \quad \checkmark
 \end{aligned}$$

TVC:

$$\begin{aligned} & \lim_{t \rightarrow \infty} \frac{\beta^t}{c_t} k_{t+1} \\ &= \lim_{t \rightarrow \infty} \frac{\beta^t}{\theta k_t^\alpha l_t^{1-\alpha} - k_{t+1}} k_{t+1} \\ &= \lim_{t \rightarrow \infty} \frac{\beta^t}{\theta k_t^\alpha l_t^{1-\alpha} - \alpha \beta \theta k_t^{\alpha-1} l_t^{1-\alpha}} \alpha \beta \theta k_t^{\alpha-1} l_t^{1-\alpha} \\ &= \lim_{t \rightarrow \infty} \frac{\beta^t}{\theta k_t^\alpha l_t^{1-\alpha} (1 - \alpha \beta)} \theta k_t^{\alpha-1} l_t^{1-\alpha} \alpha \beta \\ &= \lim_{t \rightarrow \infty} \frac{\beta^t \alpha \beta}{(1 - \alpha \beta)} = 0 \quad \checkmark \end{aligned}$$

$\Rightarrow g(k)$, $l(k)$, and $c(k)$ satisfy Euler,
intertemporal MRS + TVC 

We have guessed $V(k) = a_0 + a_1 \log k$,
solved for a_0, a_1 and the policy functions
and then verified the policy functions satisfy
our necessary (optimality) conditions.

Thus $V(k) = a_0 + a_1 \log k$ is true.