

Recitation 3 - 8105

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1 Overview

We will begin this recitation¹ reviewing the concept of Pareto Efficiency. We will then define the First Welfare Theorem and prove it. To prove the FWT, we will use multiple proofs by contradiction. This proof method reviewed, as it is important in many proofs throughout the first year economics PhD coursework. After the proof of the FWT, we briefly talk about the SPP and SWT in an endowment economy.

We will then discuss how we progress from an endowment economy to a production economy. We define an ADE and SME with production in both the infinitely lived case and OLG case.

2 Pareto Efficiency and Proof of the First Welfare Theorem

As we stated in the previous recitation, an allocation that is Pareto Efficient is both feasible and un wasteful. The following definition is for an endowment economy with I agents, but it is straightforward to extend the definition to an economy with production. Note that we will also use our usual discounted utility function in our definitions: $U_i(\{c_t^i\}_{t=0}^\infty) = \sum_{t=0}^\infty \beta^t u_i(c_t^i)$

Definition: An allocation $\{\{c_t^i\}_{t=0}^\infty\}_{i \in I}$ is **Pareto Efficient** if it is:

1.) Feasible:

- $\sum_{i \in I} c_t^i \leq \sum_{i \in I} w_t^i \quad \forall t$, and
- $c_t^i \geq 0 \quad \forall t, i$

2.) There *doesn't* exist a feasible allocation $\{\{\bar{c}_t^i\}_{t=0}^\infty\}_{i \in I}$ such that:

¹These notes are based off of Professor Tim Kehoe's lectures, Alex Wurdinger's Fall 2022 recitations, Professor Dirk Krueger's lecture notes, and past Econ 8105 TAs' (Amol Amol, Conor Ryan, Monica Tran Xuan) teaching notes. I do not claim credit for these notes and errors are my own.

- $\sum_{t=0}^{\infty} \beta^t u_i(\bar{c}_t^i) \geq \sum_{t=0}^{\infty} \beta^t u_i(c_t^i) \quad \forall i$, and
- $\sum_{t=0}^{\infty} \beta^t u_i(\bar{c}_t^i) > \sum_{t=0}^{\infty} \beta^t u_i(c_t^i)$ for at least one i

In the simple economies we have discussed thus far, without any market failures or frictions, the competitive equilibrium allocations are Pareto efficient. This is called the **First Welfare Theorem**.

First Welfare Theorem: Assume $U_i(c)$ is satisfies local non-satiation² in each arguement for each i .³ If allocation $\{\{\hat{c}_t^i\}_{t=0}^{\infty}\}_{i \in I}$ and prices $\{\hat{p}_t\}_{t=0}^{\infty}$ form a competitive equilibrium, then $\{\{\hat{c}_t^i\}_{t=0}^{\infty}\}_{i \in I}$ is Pareto efficient.

Before we move to the proof, we will review a proof strategy we will employ called **proof by contradiction**.⁴ Suppose we want to prove $P \implies Q$.⁵ In the proof of the FWT, P would be the statement " $\{\{\hat{c}_t^i\}_{t=0}^{\infty}\}_{i \in I}$ is a competitive equilibrium allocation" and Q would be " $\{\{\hat{c}_t^i\}_{t=0}^{\infty}\}_{i \in I}$ is Pareto efficient." In a proof by contradiction, we prove $P \implies Q$ by showing that $\neg(P \implies Q)$ (or equivalently $P \wedge \neg Q$)⁶ leads to a contradiction. Now, let's use this in the FWT proof.

Proof of the First Welfare Theorem: First, we assume that $U_i(c) = \sum_{t=0}^{\infty} \beta^t u_i(c_t^i)$ is strictly increasing in all of its arguments $\forall i$.

Now, to begin our proof by contradiction, assume that $\{\{\hat{c}_t^i\}_{t=0}^{\infty}\}_{i \in I}$ is a CE allocation but is not Pareto efficient. Therefore, following our definition of Pareto efficiency, there exists another feasible allocation $\{\{\bar{c}_t^i\}_{t=0}^{\infty}\}_{i \in I}$ such that:

- $\sum_{t=0}^{\infty} \beta^t u_i(\bar{c}_t^i) \geq \sum_{t=0}^{\infty} \beta^t u_i(\hat{c}_t^i) \quad \forall i$, and
- $\sum_{t=0}^{\infty} \beta^t u_j(\bar{c}_t^j) > \sum_{t=0}^{\infty} \beta^t u_j(\hat{c}_t^j)$ for some agent j

Claim 1: For agent j , the allocation $\{\bar{c}_t^j\}_{t=0}^{\infty}$ is unaffordable at equilibrium prices $\{\hat{p}_t\}_{t=0}^{\infty}$. That is:

$$\sum_{t=0}^{\infty} \hat{p}_t \bar{c}_t^j > \sum_{t=0}^{\infty} \hat{p}_t w_t^j$$

Proof of Claim 1: Suppose, towards a contradiction, that $\{\bar{c}_t^j\}_{t=0}^{\infty}$ was affordable at equilibrium prices, i.e.:

$$\sum_{t=0}^{\infty} \hat{p}_t \bar{c}_t^j \leq \sum_{t=0}^{\infty} \hat{p}_t w_t^j$$

²Or just assume a stronger condition that $U_i(c)$ is strictly increasing in all of its arguments.

³Also assume that there are a finite number of agents. For example, the FWT may not hold in certain OLG models.

⁴Angel de la Fuente's "Mathematical Methods and Models for Economists" has a nice section about proof methods and I use it as a reference here.

⁵Read as "If P then Q."

⁶Read as "P and not Q."

But then agent j was not maximizing utility in the competitive equilibrium; they chose an allocation $\{\hat{c}_t^j\}_{t=0}^\infty$ that gives them strictly less utility compared to another affordable allocation, $\{\bar{c}_t^j\}_{t=0}^\infty$. This contradicts that all households maximize utility in equilibrium. Thus it must be that $\sum_{t=0}^\infty \hat{p}_t \bar{c}_t^j > \sum_{t=0}^\infty \hat{p}_t w_t^j$. ■

Claim 2: For all agents $i \in I$, $\{\bar{c}_t^i\}_{t=0}^\infty$ is either just affordable or unaffordable at equilibrium prices $\{\hat{p}_t\}_{t=0}^\infty$, i.e.:

$$\sum_{t=0}^\infty \hat{p}_t \bar{c}_t^i \geq \sum_{t=0}^\infty \hat{p}_t w_t^i \quad \forall i$$

Proof of Claim 2: Suppose, towards a contradiction, that $\sum_{t=0}^\infty \hat{p}_t \bar{c}_t^i \geq \sum_{t=0}^\infty \hat{p}_t w_t^i \quad \forall i$ was false. This means that for some agent i , $\{\bar{c}_t^i\}_{t=0}^\infty$ is affordable and the budget constraint doesn't bind:

$$\sum_{t=0}^\infty \hat{p}_t \bar{c}_t^i < \sum_{t=0}^\infty \hat{p}_t w_t^i$$

We want to eventually show that this implies that agent i was not utility maximizing in equilibrium. To do this, we will show that we can construct another affordable allocation, $\{\tilde{c}_t^i\}_{t=0}^\infty$, that makes agent i strictly better off. This "tilde" allocation is defined as:

$$\tilde{c}_0^i = \bar{c}_0^i + \frac{\sum_{t=0}^\infty \hat{p}_t w_t^i - \sum_{t=0}^\infty \hat{p}_t \bar{c}_t^i}{\hat{p}_0}$$

$$\tilde{c}_t^i = \bar{c}_t^i \quad \forall t \geq 1$$

As a check, you can plug this "tilde" allocation into the LHS of the budget constraint and you will see that the budget constraint binds. Thus $\{\tilde{c}_t^i\}_{t=0}^\infty$ was affordable to agent i at equilibrium prices.

Clearly, $\tilde{c}_0^i > \bar{c}_0^i$ in period 0, while consumption is identical to the "bar" allocation in all other periods. Then, by our assumption that the utility function is strictly increasing in all of its arguments, this implies that:

$$\begin{aligned} \sum_{t=0}^\infty \beta^t u_i(\tilde{c}_t^i) &> \sum_{t=0}^\infty \beta^t u_i(\bar{c}_t^i) \geq \sum_{t=0}^\infty \beta^t u_i(\hat{c}_t^i) \\ \implies \sum_{t=0}^\infty \beta^t u_i(\tilde{c}_t^i) &> \sum_{t=0}^\infty \beta^t u_i(\hat{c}_t^i) \end{aligned}$$

We see that "tilde" allocation $\{\tilde{c}_t^i\}_{t=0}^\infty$ makes agent i strictly better off compared to the "hat" allocation and it was affordable at equilibrium prices. However, agent i instead chose $\{\hat{c}_t^i\}_{t=0}^\infty$ in equilibrium. This contradicts utility maximization in equilibrium and thus is contradiction. Therefore, it must be true that $\sum_{t=0}^\infty \hat{p}_t \bar{c}_t^i \geq \sum_{t=0}^\infty \hat{p}_t w_t^i \quad \forall i$. ■

Now, combine our results from claims 1 and 2:

$$\begin{aligned}\sum_{t=0}^{\infty} \hat{p}_t \bar{c}_t^j &> \sum_{t=0}^{\infty} \hat{p}_t w_t^j \quad j \in I \\ \sum_{t=0}^{\infty} \hat{p}_t \bar{c}_t^i &\geq \sum_{t=0}^{\infty} \hat{p}_t w_t^i \quad \forall i \in I\end{aligned}$$

Now, sum across all agents:⁷

$$\sum_{i \in I} \sum_{t=0}^{\infty} \hat{p}_t \bar{c}_t^i > \sum_{i \in I} \sum_{t=0}^{\infty} \hat{p}_t w_t^i$$

And rearrange the summation:

$$\sum_{t=0}^{\infty} \hat{p}_t \sum_{i \in I} \bar{c}_t^i > \sum_{t=0}^{\infty} \hat{p}_t \sum_{i \in I} w_t^i \quad (1)$$

Now, consider our market clearing condition:

$$\sum_{i \in I} \bar{c}_t^i \leq \sum_{i \in I} w_t^i$$

And multiple by $\hat{p}_t$ and sum across all time periods:⁸

$$\sum_{t=0}^{\infty} \hat{p}_t \sum_{i \in I} \bar{c}_t^i \leq \sum_{t=0}^{\infty} \hat{p}_t \sum_{i \in I} w_t^i \quad (2)$$

But we cannot have (1) and (2) simultaneously and therefore we have a contradiction. This means that $\{\{\hat{c}_t^i\}_{t=0}^{\infty}\}_{i \in I}$ must be Pareto efficient. ■

The First Welfare Theorem tells us that competitive equilibrium allocations are nested within Pareto efficient allocations (*CE allocations* $\subset$ *PE allocations*). The **Second Welfare Theorem** tells us that any Pareto efficient allocation can be supported as an equilibrium allocation, given certain prices and transfers.

Second Welfare Theorem: Assume $U_i(c)$ is continuous, concave, and strictly increasing for each i . If allocation $\{\{\hat{c}_t^i\}_{t=0}^{\infty}\}_{i \in I}$ is Pareto efficient, then there exist transfers $\{t_i\}_{i \in I}$ and prices $\{\hat{p}_t\}_{t=0}^{\infty}$ such that $\{\{\hat{c}_t^i\}_{t=0}^{\infty}\}_{i \in I}$, $\{\hat{p}_t\}_{t=0}^{\infty}$, and $\{t_i\}_{i \in I}$ form a competitive equilibrium.

⁷This is a key step for why we assume there are a finite number of agents is important in this proof. In an OLG model with an infinite number of agents, we may not be able to guarantee that the sum is finite when we sum across all agents.

⁸Again, another step where this proof may break down with infinite agents.

We can calculate Pareto efficient allocations using the **Social Planner's Problem (SPP)**. Using the SWT, we can then construct a competitive equilibrium using these PE allocations. The SPP maximizes the weighted sum (Pareto weights $\alpha_i \geq 0 \ \forall i$) of lifetime utilities subject to feasibility constraints.

Definition: Given Pareto weights $(\alpha_1, \alpha_2, \dots, \alpha_I)$, the **Social Planner's Problem** chooses $\{\hat{c}_t^1(\alpha), \hat{c}_t^2(\alpha), \dots, \hat{c}_t^I(\alpha)\}_{t=0}^\infty$ to solve:⁹

$$\begin{aligned} \max_{\{c_t^1, c_t^2, \dots, c_t^I\}_{t=0}^\infty} \quad & \sum_{i \in I} \alpha_i \sum_{t=0}^{\infty} \beta^t u_j(\hat{c}_t^i) \\ \text{s.t.} \quad & \sum_{i \in I} \hat{c}_t^i \leq \sum_{i \in I} w_t^i \quad \forall t \\ & \hat{c}_t^i \geq 0 \quad \forall t, i \end{aligned}$$

Using the SPP, we have the following proposition: If $\{\hat{c}_t^1(\alpha), \dots, \hat{c}_t^I(\alpha)\}_{t=0}^\infty$ solves the SPP, then $\{\hat{c}_t^1(\alpha), \dots, \hat{c}_t^I(\alpha)\}_{t=0}^\infty$ is Pareto efficient.

Why is this useful? Consider the two-household pure exchange economy from our previous exercises. Use the SPP to calculate a PE allocation $\{\hat{c}_t^1(\alpha), \hat{c}_t^2(\alpha)\}_{t=0}^\infty$. Then, by the SWT, we can construct an Arrow-Debreu equilibrium with transfers such that $\{\hat{c}_t^1(\alpha), \hat{c}_t^2(\alpha)\}_{t=0}^\infty$ is the equilibrium allocation. Thus, in this simple model, competitive markets can implement any PE allocation you, the social planner, desire. All you need to do is conduct a lump-sum transfer across agents and let markets operate.

The Recitation 2 notes defines an Arrow-Debreu Equilibrium with transfers, informally discusses the proof of the SWT, and also has an exercise in using Negishi's method to solve for an ADE. Please refer to those notes for more details.

⁹Let $\alpha \equiv (\alpha_1, \alpha_2, \dots, \alpha_I)$, so that we can succinctly write consumption as a function of both Pareto weights.

3 Production with Infinitely Lived Agents

In the endowment economies we have seen thus far, the households were "magically" endowed with an infinite stream of the consumption good, $\{\{w_t^i\}_{t=0}^\infty\}_{i \in I}$. These models were interesting because they showed us how households would optimally trade with each other to smooth consumption, but otherwise they seemed pretty unrealistic. To make our model a bit closer to reality, we will add production. So instead of consumption goods just popping into existence each period, a firm must actually produce them. We will add production to both the infinitely lived and OLG models.

We will start with the infinitely lived model. As in the previous recitation notes, I will describe the model environment to begin.

- **Time:** $t = 0, 1, 2, 3, \dots$
- **Agents:**
 - 1.) **Representative household**
 - * One infinitely lived household¹⁰
 - * Preferences: Household's preferences are over a sequence of consumption $c = (c_0, c_1, c_2, \dots)$ and sequence of leisure $l = (l_0, l_1, l_2, \dots)$ and are represented by:

$$U(c, l) = \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

We assume that the period utility function $u(c_t, l_t)$ is strictly increasing in both arguments, differentiable, concave, and satisfies the Inada condition.

- * Endowments: The household has the following endowments:
 - One unit¹¹ of time in each period: $\bar{l} = 1 \ \forall t$. The household uses this endowed unit of time to work (n_t) or perform leisure activities (l_t) in each period.¹²
 - $\bar{k}_0$ units of initial capital
 - $\bar{b}_0$ units of initial bonds (if in a SME)
- 2.) **Representative firm**
 - * One infinitely lived firm
 - * Chooses output (y_t^f), capital services (k_t^f), and labor (l_t^f) to maximize profits/minimize costs

¹⁰You can also think of the household as made up of an infinite number of identical consumers. This interpretation is where the term "representative household" comes from.

¹¹As Professor Kehoe mentioned in class today, the time endowment can be any positive value. For this model, we just assume a time endowment of one ($\bar{l} = 1 \ \forall t$) for simplicity.

¹²This tells us that we have a new constraint for the household; that time spent at work plus time on leisure must be less than or equal to 1 in all periods ($n_t + l_t \leq 1 \ \forall t$).

- * **Production Technology:** The firm's production function ($F(k_t^f, n_t^f)$) transforms capital services (k_t^f) and labor (n_t^f) into an output good (y_t^f):

$$y_t^f = F(k_t^f, n_t^f)$$

We assume that $F(k_t, n_t)$ exhibits constant returns to scale (CRS):¹³

$$F(\lambda k_t, \lambda n_t) = \lambda F(k_t, n_t) \quad \forall \lambda \geq 0$$

Under this assumption, we also know that firm profits will be zero in equilibrium.

Firm's output (y_t) can be consumed or invested:

$$y_t = c_t + i_t$$

This means that the firm's single production function can produce both consumption goods and investment goods. In later courses, we relax this assumption and allow different production technologies to create different types of goods.

The **law of motion on capital** is:

$$k_{t+1} = i_t + (1 - \delta)k_t$$

Which can be rearranged to:

$$i_t = k_{t+1} - (1 - \delta)k_t$$

This says that gross investment (i_t) is equal to the current capital stock (k_{t+1}) minus the amount of non-depreciated capital left from the previous period ($(1 - \delta)k_t$). $\delta \in [0, 1]$ is the depreciation rate.

- Both the household and the firm have perfect information

- **Market Structure:**

- Arrow-Debreu (markets are only open in period 0) or Sequential Markets structure (markets open each period and bonds can be traded). See previous notes for full description of these market structures.

- **Price System:**

- Prices are taken as given by both the household and firm

¹³Note that CRS also implies the following useful equation:

$$F(k_t, n_t) = F_k(k_t, n_t)k_t + F_n(k_t, n_t)n_t$$

You can verify this using the Cobb-Douglas production function: $F(k_t, n_t) = \theta k_t^\alpha n_t^{1-\alpha}$

- Wages: $\{w_t\}_{t=0}^{\infty}$
- Rental Rate on Capital: $\{r_t^k\}_{t=0}^{\infty}$
- ADE: Arrow-Debreu prices $\{p_t\}_{t=0}^{\infty}$
- SME: Interest rates on the one-period bonds $\{r_t^b\}_{t=0}^{\infty}$

When we write down the equilibrium definitions, we will assume that the household owns the capital stock and makes the investment decisions. As you have seen in Problem Set 2, we could instead assume that the firm buys the initial capital stock $\bar{k}_0$ (at a competitively determined price q_0); in this case the firm would make the investment decisions.

Let's summarize what we have so far. In all periods, the household must choose:

- c_t : How much to consume
- i_t : How much to invest in capital
- n_t : How much time to spend working
- l_t : How much time to spend on leisure activities
- k_t : How much capital services to rent out to the firm
- (If in an SME) b_{t+1} : How much bonds to borrow/lend

Note that we can simplify the decision of choosing both i_t and k_t in all periods to just choosing k_{t+1} (the $t+1$ capital stock) in all periods. This is because of the law of motion of capital ($k_{t+1} = i_t + (1-\delta)k_t$) and because $\bar{k}_0$ is given.¹⁴ You will see that I substituted $k_{t+1} - (1-\delta)k_t$ for i_t in the household's budget constraint.

Under this simplification, the household must choose the following variables in all periods: (c_t, k_{t+1}, n_t, l_t) (along with bonds b_{t+1} if in an SME).

Now we are prepared to define an ADE with production.

¹⁴To see this, note that $\hat{k}_0 = \bar{k}_0$, i.e. the household will rent out all of its endowment of capital in period 0. Thus, when the household chooses $\hat{k}_{t+1} = \hat{k}_1$ they are also implicitly choosing $\hat{i}_0$, because of the law of motion of capital: $\hat{i}_0 = \hat{k}_1 - (1-\delta)\hat{k}_0$.

Definition: An Arrow-Debreu Equilibrium (ADE) with production

consists of:

- 1.) Household Allocation: $\hat{Z}^H \equiv \{\hat{c}_t, \hat{k}_{t+1}, \hat{n}_t, \hat{l}_t\}_{t=0}^\infty$
- 2.) Firm Allocation: $\hat{Z}^F \equiv \{\hat{y}_t^f, \hat{k}_t^f, \hat{n}_t^f\}_{t=0}^\infty$
- 3.) Price System: $\hat{P} \equiv \{\hat{p}_t, \hat{r}_t^k, \hat{w}_t\}_{t=0}^\infty$

with $\hat{k}_0 = \bar{k}_0$ given, such that:

(HH) Given $\hat{P}$, the household chooses $\hat{Z}^H$ to solve:¹⁵

$$\begin{aligned} & \max_{\{c_t, k_{t+1}, n_t, l_t\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t u_i(c_t^i) \\ \text{s.t. } & \sum_{t=0}^\infty \hat{p}_t (c_t + k_{t+1} - (1 - \delta)k_t) \leq \sum_{t=0}^\infty \hat{w}_t n_t + \hat{r}_t^k k_t \\ & n_t + l_t = 1 \quad \forall t \\ & c_t, k_{t+1}, l_t, n_t \geq 0 \quad \forall t \\ & \bar{k}_0 \text{ given} \end{aligned}$$

(F) Prices are such that firms minimize costs and make zero profits:

$$\begin{aligned} \hat{y}_t^f &= F(\hat{k}_t^f, \hat{n}_t^f) \quad \forall t \\ \hat{r}_t^k &= \hat{p}_t F_k(\hat{k}_t^f, \hat{n}_t^f) \quad \forall t \\ \hat{w}_t &= \hat{p}_t F_n(\hat{k}_t^f, \hat{n}_t^f) \quad \forall t \end{aligned}$$

(MC) Market clears in each period:

$$\begin{aligned} \hat{c}_t + \hat{k}_{t+1} - (1 - \delta)\hat{k}_t &= F(\hat{k}_t, \hat{n}_t) \quad \forall t \\ \hat{k}_t &= \hat{k}_t^f \quad \forall t \\ \hat{n}_t &= \hat{n}_t^f \quad \forall t \end{aligned}$$

This completes the definition of an ADE with production. Before we move to our next equilibrium definition, a couple notes.

First, the $\hat{r}_t^k$ and $\hat{w}_t$ formulas come from the first order conditions of the firm's profit maximization/cost minimization problem.¹⁶ Since the firm does not make

¹⁵Note that we do not allow the capital stock k_{t+1} to be negative, but we implicitly allow gross investment i_t to be negative. In cases where the initial endowment of capital $\bar{k}_0$ is too high, the household may optimally choose to reduce its capital stock, which would necessitate $i_t < 0$.

¹⁶In this model, profit maximization and cost minimization result in the same choices by the firm.

investment decisions, its profit maximization problem is not dynamic. Therefore you can view the firm's problem as an infinite amount of static profit maximization problems:¹⁷

$$\begin{aligned} \max_{y_t^f, n_t^f, k_t^f} \quad & \hat{p}_t y_t^f - \hat{r}_t k_t - \hat{w}_t n_t \\ \text{s.t.} \quad & y_t^f \leq F(k_t^f, n_t^f) \\ & y_t^f, n_t^f, k_t^f \geq 0 \end{aligned}$$

Using the fact that $\hat{y}_t^f = F(\hat{k}_t^f, \hat{n}_t^f)$ at any solution, it is straightforward to take FOCs and derive the formulas for $\hat{r}_t^k$ and $\hat{w}_t$.

Second, from the law of motion of capital we know that $i_t = k_{t+1} - (1 - \delta)k_t$. Substituting this formula into the LHS of the budget constraint, you can see that the household chooses between buying c_t and i_t at price p_t in period t . Even though they are different types of goods, they are sold at the same price. This is because they come from the same output good: $y_t = c_t + i_t$.

Third, we have two new market clearing conditions:

$$\begin{aligned} \hat{k}_t &= \hat{k}_t^f \quad \forall t \\ \hat{n}_t &= \hat{n}_t^f \quad \forall t \end{aligned}$$

The first states that the amount of capital services supplied by the household equals the amount demanded by the firm. The second states that the amount of labor supplied by the household equals the amount demanded by the firm.

We will define a SME with production on the next page.

¹⁷In this case, you can use this static problem rather than the problem of maximizing the sum of profits over its lifetime e.g. $\max \sum_{t=0}^{\infty} p_t y_t^f - w_t n_t - r_t^k k_t$. However, if the firm owned the capital stock and made investment decisions, its problem would be dynamic and thus would be solved by maximizing the sum of discounted future profits.

Definition: A Sequential Markets Equilibrium (SME) with production consists of:

- 1.) Household Allocation: $\hat{Z}^H \equiv \{\hat{c}_t, \hat{k}_{t+1}, \hat{n}_t, \hat{l}_t, \hat{b}_{t+1}\}_{t=0}^\infty$
 - 2.) Firm Allocation: $\hat{Z}^F \equiv \{\hat{y}_t^f, \hat{k}_t^f, \hat{n}_t^f\}_{t=0}^\infty$
 - 3.) Price System: $\hat{P} \equiv \{\hat{r}_t^k, \hat{w}_t, \hat{r}_t^b\}_{t=0}^\infty$
- with $\hat{k}_0 = \bar{k}_0$ and $\hat{b}_0 = \bar{b}_0 = 0$ given, such that:

(HH) Given $\hat{P}$, the household chooses $\hat{Z}^H$ to solve:

$$\begin{aligned} & \max_{\{c_t, k_{t+1}, n_t, l_t, b_{t+1}\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t u_i(c_t^i) \\ \text{s.t. } & c_t + k_{t+1} - (1 - \delta)k_t + b_{t+1} \leq \hat{w}_t n_t + \hat{r}_t^k k_t + (1 + \hat{r}_t^b) b_t \quad \forall t \\ & b_t \geq -B \quad \forall t \\ & n_t + l_t = 1 \quad \forall t \\ & c_t, k_{t+1}, l_t, n_t \geq 0 \quad \forall t \\ & \bar{b}_0 = 0, \bar{k}_0 \text{ given} \end{aligned}$$

(F) Prices are such that firms minimize costs and make zero profits:

$$\begin{aligned} \hat{y}_t^f &= F(\hat{k}_t^f, \hat{n}_t^f) \quad \forall t \\ \hat{r}_t^k &= F_k(\hat{k}_t^f, \hat{n}_t^f) \quad \forall t \\ \hat{w}_t &= F_n(\hat{k}_t^f, \hat{n}_t^f) \quad \forall t \end{aligned}$$

(MC) Market clears in each period:

$$\begin{aligned} \hat{c}_t + \hat{k}_{t+1} - (1 - \delta)\hat{k}_t &= F(\hat{k}_t, \hat{n}_t) \quad \forall t \\ \hat{k}_t &= \hat{k}_t^f \quad \forall t \\ \hat{n}_t &= \hat{n}_t^f \quad \forall t \\ \hat{b}_t &= 0 \quad \forall t \end{aligned}$$

This completes the definition of a SME with production.

Note that the period t SM budget constraint can be rearranged so that k_t is only on the RHS:

$$c_t + k_{t+1} + b_{t+1} \leq \hat{w}_t n_t + (1 + \hat{r}_t^k - \delta)k_t + (1 + \hat{r}_t^b)b_t$$

We also see our new market clearing condition, $\hat{b}_t = 0 \quad \forall t$. Since there is only one household,¹⁸ there cannot be any trading of bonds in equilibrium.

¹⁸If you would rather think in terms of an infinite amount of identical consumers, $\hat{b}_t = 0 \quad \forall t$ must hold in equilibrium because all consumers make identical decisions. Since trade is impossible when everyone takes the same actions, only $\hat{b}_t = 0$ can occur in equilibrium.

4 Production in an OLG Model

I will not describe the general setup of OLG models right now. If I have time I will update these notes later in the course. The setup for production is identical to the infinitely lived model. As before, we assume that agents are endowed with one unit of time.¹⁹ We also assume the initial generation is endowed with initial capital $\bar{k}_1^0$ (and initial bond $\bar{b}_1^0 = 0$ if in a SME). I will now define an ADE and SME with production.

Definition: An ADE with production consists of:

- i.) An allocation for each generation: $(\hat{c}_1^0, \hat{n}_1^0, \hat{l}_1^0, \hat{k}_1^0), \{\hat{c}_t^t, \hat{n}_t^t, \hat{l}_t^t, \hat{k}_{t+1}^t, \hat{c}_{t+1}^t, \hat{n}_{t+1}^t, \hat{l}_{t+1}^t\}_{t=1}^\infty$
 - ii.) Firm Allocation: $\hat{Z}^F \equiv \{\hat{y}_t^f, \hat{k}_t^f, \hat{n}_t^f\}_{t=0}^\infty$
 - iii.) Price System: $P \equiv \{\hat{p}_t, \hat{r}_t^k, \hat{w}_t\}_{t=0}^\infty$
- with $\hat{k}_1^0 = \bar{k}_1^0$ and m given, such that:

- 1.) Given P , the initial generation chooses $(\hat{c}_1^0, \hat{n}_1^0, \hat{l}_1^0, \hat{k}_1^0)$ to solve:

$$\begin{aligned} & \max_{\hat{c}_1^0, \hat{n}_1^0, \hat{l}_1^0} U(\hat{c}_1^0, \hat{l}_1^0) \\ \text{s.t. } & \hat{p}_1 \hat{c}_1^0 \leq \hat{w}_1 \hat{n}_1^0 + (\hat{p}_1(1 - \delta) + \hat{r}_1^k) \bar{k}_1^0 + m \\ & \hat{n}_1^0 + \hat{l}_1^0 = 1 \\ & \hat{c}_1^0, \hat{n}_1^0, \hat{l}_1^0 \geq 0 \\ & \bar{k}_1^0 \text{ given} \end{aligned}$$

- 2.) Given P , generations $t \geq 1$ choose $Z^t \equiv (\hat{c}_t^t, \hat{n}_t^t, \hat{l}_t^t, \hat{k}_{t+1}^t, \hat{c}_{t+1}^t, \hat{n}_{t+1}^t, \hat{l}_{t+1}^t)$ to solve:

$$\begin{aligned} & \max_{Z^t} u(\hat{c}_t^t, \hat{l}_t^t) + u(\hat{c}_{t+1}^t, \hat{l}_{t+1}^t) \\ \text{s.t. } & \hat{p}_t(\hat{c}_t^t + \hat{k}_{t+1}^t) + \hat{p}_{t+1} \hat{c}_{t+1}^t \leq \hat{w}_t \hat{n}_t^t + \hat{w}_{t+1} \hat{n}_{t+1}^t + (\hat{p}_{t+1}(1 - \delta) + \hat{r}_{t+1}^k) \hat{k}_{t+1}^t \\ & \hat{n}_t^t + \hat{l}_t^t = 1 \\ & \hat{n}_{t+1}^t + \hat{l}_{t+1}^t = 1 \\ & \hat{c}_t^t, \hat{n}_t^t, \hat{l}_t^t, \hat{c}_{t+1}^t, \hat{n}_{t+1}^t, \hat{l}_{t+1}^t, \hat{k}_{t+1}^t \geq 0 \end{aligned}$$

- 3.) Prices are such that firms minimize costs and make zero profits:

$$\hat{y}_t^f = F(\hat{k}_t^f, \hat{n}_t^f) \quad \forall t$$

¹⁹You could relax this assumption by assuming that each young generation gets a time endowment of $\bar{l}_1$ and each old generation gets a time endowment of $\bar{l}_2$. For example, you could say that old generations have less time to spend on work and on leisure activities compared to the young generations (maybe this is because the old generations sleep more!). Thus you would pick time endowments that satisfy $\bar{l}_2 < \bar{l}_1$ in each period.

$$\begin{aligned}\hat{r}_t^k &= \hat{p}_t F_k(\hat{k}_t^f, \hat{n}_t^f) \quad \forall t \\ \hat{w}_t &= \hat{p}_t F_n(\hat{k}_t^f, \hat{n}_t^f) \quad \forall t\end{aligned}$$

4.) The markets clear $\forall t$

$$\begin{aligned}\hat{c}_t^t + \hat{c}_t^{t-1} + \hat{k}_{t+1}^t - (1 - \delta)\hat{k}_t^{t-1} &= F(\hat{k}_t^f, \hat{n}_t^f) \\ \hat{k}_t^{t-1} &= \hat{k}_t^f \quad \forall t \\ \hat{n}_t^t + \hat{n}_t^{t-1} &= \hat{n}_t^f \quad \forall t\end{aligned}$$

Now, I will define a Sequential Markets Equilibrium with production.

Definition: A SME with production consists of:

- i.) An allocation for each generation: $(\hat{c}_1^0, \hat{n}_1^0, \hat{l}_1^0, \hat{k}_1^0), \{\hat{c}_t^t, \hat{n}_t^t, \hat{l}_t^t, \hat{k}_{t+1}^t, \hat{b}_{t+1}^t, \hat{c}_{t+1}^t, \hat{n}_{t+1}^t, \hat{l}_{t+1}^t\}_{t=1}^\infty$
 - ii.) Firm Allocation: $\hat{Z}^F \equiv \{\hat{y}_t^f, \hat{k}_t^f, \hat{n}_t^f\}_{t=0}^\infty$
 - iii.) Price System: $P \equiv \{\hat{r}_t^b, \hat{r}_t^k, \hat{w}_t\}_{t=0}^\infty$
- with $\hat{k}_1^0 = \bar{k}_1^0, \hat{b}_1^0 = \bar{b}_1^0 = 0$ and m given, such that:

- 1.) Given P , the initial generation chooses $(\hat{c}_1^0, \hat{n}_1^0, \hat{l}_1^0, \hat{k}_1^0)$ to solve:

$$\begin{aligned} & \max_{\hat{c}_1^0, \hat{n}_1^0, \hat{l}_1^0} U(\hat{c}_1^0, \hat{l}_1^0) \\ \text{s.t. } & \hat{c}_1^0 \leq \hat{w}_1 \hat{n}_1^0 + (1 + \hat{r}_1^k - \delta) \bar{k}_1^0 + m \\ & \hat{n}_1^0 + \hat{l}_1^0 = 1 \\ & \hat{c}_1^0, \hat{n}_1^0, \hat{l}_1^0 \geq 0 \\ & \bar{b}_1^0 = 0, \bar{k}_1^0 \text{ given} \end{aligned}$$

- 2.) Given P , generations $t \geq 1$ choose $Z^t \equiv (\hat{c}_t^t, \hat{n}_t^t, \hat{l}_t^t, \hat{k}_{t+1}^t, \hat{b}_{t+1}^t, \hat{c}_{t+1}^t, \hat{n}_{t+1}^t, \hat{l}_{t+1}^t)$ to solve:

$$\begin{aligned} & \max_{Z^t} u(\hat{c}_t^t, \hat{l}_t^t) + u(\hat{c}_{t+1}^t, \hat{l}_{t+1}^t) \\ \text{s.t. } & \hat{c}_t^t + \hat{k}_{t+1}^t + \hat{b}_{t+1}^t \leq \hat{w}_t \hat{n}_t^t \\ & \hat{c}_{t+1}^t \leq \hat{w}_{t+1} \hat{n}_{t+1}^t + (1 + \hat{r}_{t+1}^k - \delta) \hat{k}_{t+1}^t + (1 + \hat{r}_{t+1}^b) \hat{b}_{t+1}^t \\ & \hat{n}_t^t + \hat{l}_t^t = 1 \\ & \hat{n}_{t+1}^t + \hat{l}_{t+1}^t = 1 \\ & \hat{c}_t^t, \hat{n}_t^t, \hat{l}_t^t, \hat{c}_{t+1}^t, \hat{n}_{t+1}^t, \hat{l}_{t+1}^t, \hat{k}_{t+1}^t \geq 0 \end{aligned}$$

- 3.) Prices are such that firms minimize costs and make zero profits:

$$\begin{aligned} \hat{y}_t^f &= F(\hat{k}_t^f, \hat{n}_t^f) \quad \forall t \\ \hat{r}_t^k &= F_k(\hat{k}_t^f, \hat{n}_t^f) \quad \forall t \\ \hat{w}_t &= F_n(\hat{k}_t^f, \hat{n}_t^f) \quad \forall t \end{aligned}$$

- 4.) Market clears $\forall t$:

$$\begin{aligned} \hat{c}_t^t + \hat{c}_t^{t-1} + \hat{k}_{t+1}^t - (1 - \delta) \hat{k}_t^{t-1} &= F(\hat{k}_t^f, \hat{n}_t^f) \\ \hat{k}_t^{t-1} &= \hat{k}_t^f \quad \forall t \\ \hat{n}_t^t + \hat{n}_t^{t-1} &= \hat{n}_t^f \quad \forall t \\ \hat{b}_2^1 &= m \\ \hat{b}_{t+1}^t &= \left[\prod_{\tau=2}^t (1 + \hat{r}_\tau^b) \right] m \quad \forall t \geq 2 \end{aligned}$$