

Macro Theory (8105) Recitation 2

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1 Overview

In this set of notes,¹ I define, characterize, and calculate a Sequential Markets Equilibrium. I mention the ADE and SME equivalence result. I then cover Pareto Efficiency, the Social Planner's Problem, the Welfare Theorems, and ADE with transfers. We finish with an exercise in using Negishi's method to calculate an ADE.

Appendix A discusses transversality conditions.

2 Sequential Markets Equilibrium

In our first recitation, we defined and characterized an Arrow-Debreu equilibrium. In this class we will now define another type of competitive equilibrium, a **Sequential Markets Equilibrium (SME)**. The environment for an SME only differs from an ADE in the market structure and price system. For completeness, I present a complete description of the environment:

- **Time:** $t = 0, 1, 2, 3, \dots$
- **Commodities:** One nonstorable consumption good per period
- **Agents:**
 - Two infinitely lived households $i \in I = \{1, 2\}$
 - Preferences over a sequence of consumption $c^i = (c_0^i, c_1^i, c_2^i, \dots)$ are represented by:

$$U_i(c^i) = \sum_{t=0}^{\infty} \beta_i^t u_i(c_t^i)$$

¹These notes are based off of Professor Tim Kehoe's lectures, Alex Wurdinger's Fall 2022 recitations, Professor Dirk Krueger's lecture notes, Professor Wouter den Haan's teaching notes, past Econ 8105 TAs' (Amol Amol, Conor Ryan, Monica Tran Xuan), and Bipul Verma's 8102 recitation notes. I do not claim credit for these notes and errors are my own.

where $\beta_i \in (0, 1)$ is the discount factor.²

- Each agent has an endowment $\{w_t^i\}_{t=0}^\infty = \{w_0^i, w_1^i, w_2^i, \dots\}$, where w_t^i denotes the amount of the consumption good endowed to agent i in period t
- Both agents have perfect information

• **Market Structure:**

- Markets are open each period
- Agents exchange one period bonds with each other
- Bonds are riskless

• **Price System:**

- Interest rates on the one-period bonds: $\{r_{t+1}\}_{t=0}^\infty$
- If an agent borrows b_{t+1}^i in period t , they must repay $(1 + r_{t+1})b_{t+1}^i$ in period $t+1$

We are now prepared to define a SME.

Definition: A **Sequential Markets Equilibrium** consists of:

- 1.) Household allocations: $\hat{Z}^i \equiv \{\hat{c}_t^i, \hat{b}_{t+1}^i\}_{t=0}^\infty \forall i \in \{1, 2\}$
 - 2.) Interest rates: $\hat{P} \equiv \{\hat{r}_{t+1}\}_{t=0}^\infty$ such that:
- (HH) Given $\hat{P}$, household i chooses $\hat{Z}^i$ to solve:

$$\begin{aligned} \max_{\{c_t^i, b_{t+1}^i\}_{t=0}^\infty} & \sum_{t=0}^\infty \beta^t u_i(c_t^i) \\ \text{s.t.} \quad & c_t^i + b_{t+1}^i \leq (1 + \hat{r}_t)b_t^i + w_t^i \quad \forall t \\ & b_{t+1}^i \geq -B \quad \forall t \\ & b_0^i = \bar{b}_i = 0 \quad \text{given} \\ & c_t^i \geq 0 \quad \forall t \end{aligned}$$

(MC) The goods market and bond market clears in each period:

$$\begin{aligned} \sum_{i \in I} \hat{c}_t^i &= \sum_{i \in I} w_t^i \quad \forall t \\ \sum_{i \in I} \hat{b}_{t+1}^i &= 0 \quad \forall t \end{aligned}$$

This completes our definition of an SME.

²Note that β_i indicates that the discount factor can be different across households. In most of our exercises, however, households will have the same discount factor $\beta_i = \beta \forall i \in I$.

A key component of the SME is the no Ponzi condition $b_{t+1}^i \geq -B$. As Professor Kehoe mentioned in lecture, this constraint prevents the agents from running a Ponzi scheme.³

Note that we now have another market clearing condition; the bond market must clear. In other words, total bonds must sum to zero within each period. Intuitively, this means that if agent 1 borrows $b = b_t^1 > 0$ in period t , agent 2 must be lending out $-b = b_t^2 < 0$ in equilibrium.

3 Characterizing a SME

To begin characterizing the SME, we will write down the Lagrangian for each household's maximization problem.⁴

$$\mathcal{L}^i = \sum_{t=0}^{\infty} \beta^t u_i(c_t^i) + \sum_{t=0}^{\infty} \mu_t^i ((1 + \hat{r}_t) b_t^i + w_t^i - c_t^i - b_{t+1}^i) + \sum_{t=0}^{\infty} \eta_t^i (b_{t+1}^i + B) + \sum_{t=0}^{\infty} \theta_t^i c_t^i$$

The KKT FOCs are:

$$\begin{array}{ll} \frac{\partial \mathcal{L}^i}{\partial c_t^i} = 0 & : \quad \beta^t u_i'(\hat{c}_t^i) - \mu_t^i + \theta_t^i = 0 \\ \frac{\partial \mathcal{L}^i}{\partial b_{t+1}^i} = 0 & : \quad -\mu_t^i + \mu_{t+1}^i (1 + \hat{r}_{t+1}) + \eta_t^i = 0 \\ \frac{\partial \mathcal{L}^i}{\partial \mu_t^i} \geq 0 & : \quad (1 + \hat{r}_t) \hat{b}_t^i + w_t^i - \hat{c}_t^i - \hat{b}_{t+1}^i \geq 0 \\ \mu_t^i \geq 0 & : \quad \mu_t^i \geq 0 \\ \mu_t^i \frac{\partial \mathcal{L}^i}{\partial \mu_t^i} = 0 & : \quad \mu_t^i ((1 + \hat{r}_t) \hat{b}_t^i + w_t^i - \hat{c}_t^i - \hat{b}_{t+1}^i) = 0 \\ \frac{\partial \mathcal{L}^i}{\partial \eta_t^i} \geq 0 & : \quad \hat{b}_{t+1}^i + B \geq 0 \\ \eta_t^i \geq 0 & : \quad \eta_t^i \geq 0 \\ \eta_t^i \frac{\partial \mathcal{L}^i}{\partial \eta_t^i} = 0 & : \quad \eta_t^i (\hat{b}_{t+1}^i + B) = 0 \\ \frac{\partial \mathcal{L}^i}{\partial \theta_t^i} \geq 0 & : \quad \hat{c}_t^i \geq 0 \\ \theta_t^i \geq 0 & : \quad \theta_t^i \geq 0 \\ \theta_t^i \frac{\partial \mathcal{L}^i}{\partial \theta_t^i} = 0 & : \quad \theta_t^i \hat{c}_t^i = 0 \end{array}$$

³As a technical note, in this model we want B to be large enough so that this constraint never binds.

⁴Again assuming that the utility functions are differentiable.

If we assume that $u_i(c)$ satisfies $\lim_{c \rightarrow 0} u'_i(c) = +\infty$ and that B is large enough so that the no Ponzi condition is never binding, we know that $\theta_t^i = 0$ and $\eta_t^i = 0$. If we further assume that $u_i(c)$ is strictly increasing, we know that the sequential markets budget constraint will bind each period. Under these assumptions, we can rewrite the KKT FOCs as:

$$\begin{aligned}\beta^t u'_i(\hat{c}_t^i) &= \mu_t^i \\ \frac{\mu_t^i}{\mu_{t+1}^i} &= (1 + \hat{r}_{t+1}) \\ \hat{c}_t^i + \hat{b}_{t+1}^i &= (1 + \hat{r}_t) \hat{b}_t^i + w_t^i \\ \hat{b}_{t+1}^i &> -B\end{aligned}$$

As in the ADE case, we also need the market clearing conditions to characterize the equilibrium:

$$\begin{aligned}\sum_{i \in I} \hat{c}_t^i &= \sum_{i \in I} w_t^i \quad \forall t \\ \sum_{i \in I} \hat{b}_{t+1}^i &= 0 \quad \forall t\end{aligned}$$

We need two more conditions to fully characterize this SME. The first is the period zero bond endowed to the households: $\hat{b}_0^i = \bar{b}_i$. The second, the **transversality condition (TVC)**,⁵ guarantees that households do not save too much over time.⁶ In this model, the TVC is written as:

$$\lim_{t \rightarrow \infty} \mu_t^i \hat{b}_{t+1}^i = 0$$

Which is equivalent to:

$$\lim_{t \rightarrow \infty} \beta^t u'_i(\hat{c}_t^i) \hat{b}_{t+1}^i = 0$$

We have now fully characterized our SME.

4 Calculating a SME

As I mentioned in the previous set of notes, given the same endowments, an ADE and SME have the same optimal consumption allocations. Therefore, if we have the same endowments as in the ADE exercise (i.e. $\{w_t^1\}_{t=0}^\infty = \{a, 0, a, 0, \dots\}$ and $\{w_t^2\}_{t=0}^\infty = \{0, a, 0, a, \dots\}$), the SME consumption allocation will be identical to the ADE allocation:

$$\hat{c}_t^1 = \frac{a}{1 + \beta} \quad \forall t$$

⁵See the appendix for more details about the TVC. It is not required knowledge at this point in the class, so only read it if you are interested.

⁶In this regard, its purpose is the opposite of the no Ponzi condition.

$$\hat{c}_t^2 = \frac{a\beta}{1+\beta} \quad \forall t$$

Now we just need to calculate interest rates and bonds. Lets begin with interest rates. From the consumption FOCs and bond FOCs, we can derive the Euler equation:

$$\frac{u'_i(\hat{c}_t^i)}{\beta u'_i(\hat{c}_{t+1}^i)} = 1 + \hat{r}_{t+1} \quad \forall t$$

Remember the Euler equation in the ADE:

$$\frac{u'_i(\hat{c}_t^i)}{\beta u'_i(\hat{c}_{t+1}^i)} = \frac{\hat{p}_t}{\hat{p}_{t+1}}$$

We already calculated prices in an ADE, so you can solve for interest rates in the SME using the Euler equations above.

To calculate bonds, combine the allocations and prices you have already solved for with the sequential markets budget constraint:

$$\hat{c}_t^i + \hat{b}_{t+1}^i = (1 + \hat{r}_t)\hat{b}_t^i + w_t^i$$

Once you have calculated the bonds for household i , $\{\hat{b}_{t+1}^i\}_{t=0}^\infty$, you can use the bond market clearing condition to solve for the other household's bonds:

$$\hat{b}_{t+1}^1 + \hat{b}_{t+1}^2 = 0 \iff \hat{b}_{t+1}^1 = -\hat{b}_{t+1}^2 \quad \forall t$$

5 Pareto Efficiency, Welfare Theorems, and Negishi's Method

An allocation that is **Pareto Efficient** is both feasible and un wasteful. Feasibility fits our usual definition - the amount consumed in each period is less than or equal to the amount available (endowed or produced) in that period (and is positive). By saying an allocation is "un wasteful," I mean that no other allocation would make at least one agent strictly better off while keeping all other agents just as well off. Now we will define Pareto efficiency in an endowment economy with I households who are infinitely lived.

Definition: An allocation $\{\{c_t^i\}_{t=0}^\infty\}_{i \in I}$ is **Pareto Efficient** if it is:

1.) Feasible:

- $\sum_{i \in I} c_t^i \leq \sum_{i \in I} w_t^i \quad \forall t$, and
- $c_t^i \geq 0 \quad \forall t, i$

2.) There *doesn't* exist a feasible allocation $\{\{\bar{c}_t^i\}_{t=0}^\infty\}_{i \in I}$ such that:

- $U_i(\{\bar{c}_t^i\}_{t=0}^\infty) \geq U_i(\{c_t^i\}_{t=0}^\infty) \quad \forall i$, and

- $U_i(\{\hat{c}_t^i\}_{t=0}^\infty) > U_i(\{c_t^i\}_{t=0}^\infty)$ for at least one i

In the simple economies we have discussed thus far, without any market failures or frictions, the competitive equilibrium allocations are Pareto efficient. This is called the **First Welfare Theorem**.

First Welfare Theorem: Assume $U_i(c)$ is satisfies local non-satiation for each i .⁷ If allocation $\{\{\hat{c}_t^i\}_{t=0}^\infty\}_{i \in I}$ and prices $\{\hat{p}_t\}_{t=0}^\infty$ form a competitive equilibrium, then $\{\{\hat{c}_t^i\}_{t=0}^\infty\}_{i \in I}$ is Pareto efficient.

The First Welfare Theorem tells us that competitive equilibrium allocations are nested within Pareto efficient allocations ($CE \subset PE$). The **Second Welfare Theorem** tells us that any Pareto efficient allocation can be supported as an equilibrium allocation, given certain prices and transfers.

Second Welfare Theorem: Assume $U_i(c)$ is continuous, concave, and strictly increasing for each i . If allocation $\{\{\hat{c}_t^i\}_{t=0}^\infty\}_{i \in I}$ is Pareto efficient, then there exist transfers $\{t_i\}_{i \in I}$ and prices $\{\hat{p}_t\}_{t=0}^\infty$ such that $\{\{\hat{c}_t^i\}_{t=0}^\infty\}_{i \in I}$, $\{\hat{p}_t\}_{t=0}^\infty$, and $\{t_i\}_{i \in I}$ form a competitive equilibrium.

We can calculate Pareto efficient allocations using the **Social Planner's Problem (SPP)**. Using the SWT, we can then construct a competitive equilibrium using these PE allocations. The SPP maximizes the weighted sum (Pareto weights $\alpha_i \geq 0 \ \forall i$) of lifetime utilities subject to feasibility constraints.

Definition: Given Pareto weights $(\alpha_1, \alpha_2, \dots, \alpha_I)$, the **Social Planner's Problem** chooses $\{\hat{c}_t^1(\alpha), \hat{c}_t^2(\alpha), \dots, \hat{c}_t^I(\alpha)\}_{t=0}^\infty$ to solve:⁸

$$\begin{aligned} \max_{\{c_t^1, c_t^2, \dots, c_t^I\}_{t=0}^\infty} \quad & \sum_{i \in I} \alpha_i U_i(\{c_t^i\}_{t=0}^\infty) \\ \text{s.t.} \quad & \sum_{i \in I} c_t^i \leq \sum_{i \in I} w_t^i \quad \forall t \\ & c_t^i \geq 0 \quad \forall t, i \end{aligned}$$

Using the SPP, we have the following proposition: If $\{\hat{c}_t^1(\alpha), \dots, \hat{c}_t^I(\alpha)\}_{t=0}^\infty$ solves the SPP, then $\{\hat{c}_t^1(\alpha), \dots, \hat{c}_t^I(\alpha)\}_{t=0}^\infty$ is Pareto efficient.

Why is this useful? Consider the two-household pure exchange economy from our previous exercises. Use the SPP to calculate a PE allocation $\{\hat{c}_t^1(\alpha), \hat{c}_t^2(\alpha)\}_{t=0}^\infty$. Then, by the SWT, we can construct an Arrow-Debreu equilibrium with transfers such that $\{\hat{c}_t^1(\alpha), \hat{c}_t^2(\alpha)\}_{t=0}^\infty$ is the equilibrium allocation. Thus, in this

⁷Also assume that there are a finite number of agents. For example, the FWT may not hold in certain OLG models.

⁸Let $\alpha \equiv (\alpha_1, \alpha_2, \dots, \alpha_I)$, so that we can succinctly write consumption as a function of both Pareto weights.

simple model, competitive markets can implement any PE allocation you, the social planner, desire. All you need to do is conduct a lump-sum transfer across agents and let markets operate.

Definition: An **Arrow-Debreu Equilibrium with transfers** consists of:

- 1.) Household Allocations: $\hat{Z}^i \equiv \{\hat{c}_t^i\}_{t=0}^\infty \forall i \in \{1, 2\}$
- 2.) Price System: $\hat{P} \equiv \{\hat{p}_t\}_{t=0}^\infty$
- 3.) Transfers: $\hat{t}_i \forall i \in \{1, 2\}$ such that:

(HH) Given $\hat{P}$, household $i \in \{1, 2\}$ chooses $\hat{Z}^i$ to solve:

$$\begin{aligned} & \max_{\{c_t^i\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t u_i(c_t^i) \\ s.t. \quad & \sum_{t=0}^\infty \hat{p}_t c_t^i \leq \sum_{t=0}^\infty \hat{p}_t w_t^i + \hat{t}_i \\ & c_t^i \geq 0 \quad \forall t \end{aligned}$$

(MC) Market clears in each period:

$$\sum_{i \in I} \hat{c}_t^i = \sum_{i \in I} w_t^i \quad \forall t$$

It should be clear that if $t_i = 0$ for all households, we have our normal ADE.

Let's see how we can use the SWT. Assume we have the two households from our previous examples. We know that the aggregate endowment in each period is $w_t^1 + w_t^2 = a \quad \forall t$. The weights α_1 and α_2 are non-negative and given.⁹ The social planner chooses $\{\hat{c}_t^1(\alpha), \hat{c}_t^2(\alpha)\}_{t=0}^\infty$ to solve:

$$\begin{aligned} & \max_{\{c_t^1, c_t^2\}_{t=0}^\infty} \alpha_1 \sum_{t=0}^\infty \beta^t \ln(c_t^1) + \alpha_2 \sum_{t=0}^\infty \beta^t \ln(c_t^2) \\ s.t. \quad & c_t^1 + c_t^2 \leq w_t^1 + w_t^2 = a \quad \forall t \\ & c_t^i \geq 0 \quad \forall t, i \end{aligned}$$

Let μ_t be the Lagrange multiplier on the feasibility condition at time t . The consumption FOCs are:

$$\alpha_i \frac{\beta^t}{\hat{c}_t^i} = \mu_t \quad \forall t, i$$

⁹In the case that one of the weights is equal to zero, the social planner will give the total endowment to the household with the non-zero weight. The household with $\alpha_i = 0$ will consume nothing.

If divide this FOC at time t for household 1 by the FOC at time t for household 2, we get:

$$\frac{\alpha_1 \hat{c}_t^2}{\alpha_2 \hat{c}_t^1} = 1 \iff \frac{\hat{c}_t^2}{\hat{c}_t^1} = \frac{\alpha_2}{\alpha_1} \iff \hat{c}_t^2 = \frac{\alpha_2}{\alpha_1} \hat{c}_t^1$$

Plugging this into the feasibility condition, we can solve for the solution to the SPP; Pareto efficient allocations that are a function of the α weights:

$$\hat{c}_t^1(\alpha) = \frac{a}{1 + \frac{\alpha_2}{\alpha_1}} = \frac{\alpha_1}{\alpha_2 + \alpha_1} a \quad \forall t$$

$$\hat{c}_t^2(\alpha) = \frac{a}{1 + \frac{\alpha_1}{\alpha_2}} = \frac{\alpha_2}{\alpha_2 + \alpha_1} a \quad \forall t$$

We see that both households consume a constant share of the aggregate endowment across time, and their share is dependent on the relative value of their Pareto weight α_i . You, the social planner, can choose any non-negative values of Pareto weights (α_1, α_2) and these allocations will remain Pareto efficient.

Given a PE allocation, $Z^{PE} \equiv \{\hat{c}_t^1(\alpha), \hat{c}_t^2(\alpha)\}_{t=0}^\infty$, how can we can construct an ADE with transfers such that the equilibrium allocation is identical to the PE allocation? An ADE allocation must be feasible in all periods and must satisfy the optimality conditions from the households' optimization problem (including the budget constraints).

We know that any PE allocation Z^{PE} will satisfy feasibility (by definition). Since feasibility is not an issue, we will focus on the optimality conditions. Using the fact that these utility functions are strictly concave, we know that the KKT FOCs are sufficient for solving the households' maximization problems. I.e. any prices and allocations (along with Lagrange multipliers) that satisfy the corresponding optimality conditions are a solution to the household maximization problem. Consider the consumption FOCs in an ADE with transfers:¹⁰

$$\frac{\beta^t}{\hat{c}_t^i} = \lambda_i \hat{p}_t \quad \forall t, i \quad (\star)$$

If we can find values of $\hat{p}_t$, $\hat{c}_t^i$, and λ_i such that:

- i.) $\frac{\beta^t}{\hat{c}_t^i} = \lambda_i \hat{p}_t \quad \forall t, i$, and
 - ii.) The budget constraints for both households are satisfied,
- then we have found the equilibrium allocation.

Now, look at the consumption FOCs from the SPP that Z^{PE} solves:

$$\alpha_i \frac{\beta^t}{\hat{c}_t^i} = \mu_t \quad \forall t, i$$

¹⁰Note that these consumption FOCs are identical to those in a normal ADE (without transfers)

Clearly, if we set:

$$\lambda_i = \frac{1}{\alpha_i} \quad \forall i$$

$$\hat{p}_t = \mu_t \quad \forall t$$

Then $(\star)$ is satisfied when we use $Z^{PE} \equiv \{\hat{c}_t^1(\alpha), \hat{c}_t^2(\alpha)\}_{t=0}^\infty$ as the allocation. Now we just need to verify that our prices and allocation satisfy the budget constraints. To ensure both households can afford their PE allocation, we need to add transfers to the budget constraints. A transfer $\hat{t}_i(\alpha)$ is a lump-sum amount (denominated in period 0 consumption good) such that Z^{PE} is affordable for both households at prices $\hat{p}_t = \mu_t$ in the CE.¹¹

$$\hat{t}_i(\alpha) = \sum_{t=0}^{\infty} \mu_t (\hat{c}_t^i(\alpha) - w_t^i)$$

Using the transfers, we can guarantee that the budget constraints are satisfied for both households:

$$\sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^i(\alpha) = \sum_{t=0}^{\infty} \hat{p}_t w_t^i + \hat{t}_i \quad \forall i$$

Therefore, we have found prices $\{\hat{p}_t\}_{t=0}^\infty$, an allocation $\{\hat{c}_t^1(\alpha), \hat{c}_t^2(\alpha)\}_{t=0}^\infty$, and transfers $\{\hat{t}_i(\alpha)\}_{i \in \{1,2\}}$ that satisfy feasibility, the consumption FOCs, and the budget constraints. We have constructed a CE that implements our PE allocation as the equilibrium allocation!

We can use similar logic to calculate a normal ADE (without transfers) using just the SPP and transfer function. In **Negishi's Method**, we use PE allocations¹² $\{\hat{c}_t^1(\alpha), \hat{c}_t^2(\alpha)\}_{t=0}^\infty$ and the transfer function to solve for the Pareto weights $\hat{\alpha}$ such that transfers are 0 ($t_i = 0 \quad \forall i$). We then use these weights to calculate a new PE allocation $\{\hat{c}_t^1(\hat{\alpha}), \hat{c}_t^2(\hat{\alpha})\}_{t=0}^\infty$ and apply the SWT to get our ADE (with transfers = 0).

Let's see the Negishi method in action. Use the SPP consumption FOC for household i and our formulas for $\hat{c}_t^i(\alpha)$ to solve for μ_t :

$$\mu_t = \alpha_i \frac{\beta^t}{\hat{c}_t^i} = \alpha_i \frac{\beta^t}{\frac{\alpha_i}{\alpha_2 + \alpha_1} a} = (\alpha_2 + \alpha_1) \frac{1}{a} \beta^t$$

Now plug this into our transfer function:

$$t_i(\alpha) = \sum_{t=0}^{\infty} (\alpha_2 + \alpha_1) \frac{1}{a} \beta^t (\hat{c}_t^i(\alpha) - w_t^i)$$

¹¹In problem set 1, you verify that $t_1 + t_2 = 0$. Intuitively, if household 1 gets a positive transfer payment ($t_1 > 0$), household 2 is "taxed" the same amount ($t_2 = -t_1 < 0$). You are "transferring" wealth from household 2 to household 1.

¹²Computed using the SPP with arbitrary Pareto weights.

We can normalize $\alpha_2 + \alpha_1 = 1$ since the transfer function is homogeneous of degree 1. We will now set $t_1(\hat{\alpha}) = 0$ and solve for $\hat{\alpha}$:

$$\begin{aligned} t_1(\hat{\alpha}) &= \sum_{t=0}^{\infty} \frac{1}{a} \beta^t (\hat{\alpha}_1 a - w_t^i) = 0 \\ \implies \frac{\hat{\alpha}_1}{1 - \beta} &= \frac{1}{1 - \beta^2} \\ \implies \hat{\alpha}_1 &= \frac{1}{1 + \beta} \\ \implies \hat{\alpha}_2 &= 1 - \hat{\alpha}_1 = \frac{\beta}{1 + \beta} \end{aligned}$$

Plugging these new Pareto weights into the PE consumption equations, we get:

$$\begin{aligned} \hat{c}_t^1(\hat{\alpha}) &= \frac{a}{1 + \beta} \quad \forall t \\ \hat{c}_t^2(\hat{\alpha}) &= \frac{a\beta}{1 + \beta} \quad \forall t \end{aligned}$$

These are exactly the same allocations as we calculated for the ADE in the Recitation 1 notes. Prices are also the same, up to a scalar multiple:¹³

$$\hat{p}_t = \mu_t = \frac{\beta^t}{a}$$

We have calculated an ADE using just the SPP and transfer function. That is pretty neat if you ask me!

The Negishi method is useful for computing CE since the SPP is a much simpler optimization problem compared to solving for an equilibrium. The SPP is just a single constrained maximization problem, while solving a CE involves multiple constrained maximization problems, market clearing conditions, and prices.

¹³Since relative prices are all that matter to the households, the scalar multiple is irrelevant. As discussed in class, I could multiply all prices by $a > 0$ and the household maximization problem would remain unchanged.

6 Appendix A: Transversality Conditions

You do not need to know these details right now. I post it solely for those interested.

One way to think about the TVC is as a "terminal condition" for an asset.¹⁴ For example, in a sequential markets model with finite periods (T periods), the amount borrowed/lent in final period must be greater than or equal to zero in equilibrium ($\hat{b}_{T+1}^i \geq 0$).

Suppose both households only live until period T . We rewrite their maximization problem as:

$$\begin{aligned} & \max_{\{c_t^i, b_{t+1}^i\}_{t=0}^T} \sum_{t=0}^T \beta^t u_i(c_t^i) \\ \text{s.t. } & c_t^i + b_{t+1}^i \leq (1 + \hat{r}_t)b_t^i + w_t^i \quad \forall t \in \{0, 1, \dots, T\} \\ & b_{t+1}^i \geq -B \quad \forall t \in \{0, 1, \dots, T-1\} \\ & c_t^i \geq 0 \quad \forall t \in \{0, 1, \dots, T\} \\ & b_0^i = \bar{b}_i = 0 \text{ given} \\ & b_{T+1}^i \geq 0 \end{aligned}$$

Note the new constraint: $b_{T+1}^i \geq 0$. This prevents the households from borrowing a large amount in the last period and thus never repaying their debt.

The Lagrangian for this problem is:

$$\mathcal{L}^i = \sum_{t=0}^T \beta^t u_i(c_t^i) + \sum_{t=0}^T \mu_t^i ((1 + \hat{r}_t)b_t^i + w_t^i - c_t^i - b_{t+1}^i) + \sum_{t=0}^T \eta_t^i (b_{t+1}^i + B) + \sum_{t=0}^T \theta_t^i c_t^i + \psi_i b_{T+1}^i$$

¹⁴Depending on the model, there can be multiple TVCs.

Consider the KKT FOCs in the final period T :

$$\begin{array}{ll}
\frac{\partial \mathcal{L}^i}{\partial \hat{c}_T^i} = 0 & : \quad \beta^T u'_i(\hat{c}_T^i) - \mu_T^i + \theta_T^i = 0 \\
\frac{\partial \mathcal{L}^i}{\partial \hat{b}_{T+1}^i} = 0 & : \quad -\mu_T^i + \psi_i = 0 \\
\frac{\partial \mathcal{L}^i}{\partial \mu_T^i} \geq 0 & : \quad (1 + \hat{r}_T) \hat{b}_T^i + w_T^i - \hat{c}_T^i - \hat{b}_{T+1}^i \geq 0 \\
\mu_T^i \geq 0 & : \quad \mu_T^i \geq 0 \\
\mu_T^i \frac{\partial \mathcal{L}^i}{\partial \mu_T^i} = 0 & : \quad \mu_T^i \left((1 + \hat{r}_T) \hat{b}_T^i + w_T^i - \hat{c}_T^i - \hat{b}_{T+1}^i \right) = 0 \\
\frac{\partial \mathcal{L}^i}{\partial \theta_T^i} \geq 0 & : \quad \hat{c}_T^i \geq 0 \\
\theta_T^i \geq 0 & : \quad \theta_T^i \geq 0 \\
\theta_T^i \frac{\partial \mathcal{L}^i}{\partial \theta_T^i} = 0 & : \quad \theta_T^i \hat{c}_T^i = 0 \\
\frac{\partial \mathcal{L}^i}{\partial \psi_i} \geq 0 & : \quad \hat{b}_{T+1}^i \geq 0 \\
\psi_i \geq 0 & : \quad \psi_i \geq 0 \\
\psi_i \frac{\partial \mathcal{L}^i}{\partial \psi_i} = 0 & : \quad \psi_i \hat{b}_{T+1}^i = 0
\end{array}$$

Again, assume that $u_i(c)$ satisfies $\lim_{c \rightarrow 0} u'_i(c) = +\infty$ so that optimal consumption is positive and therefore $\theta_T^i = 0$. We will use the first two conditions and the last condition from above:

$$\beta^T u'_i(\hat{c}_T^i) = \mu_T^i$$

$$\mu_T^i = \psi_i$$

$$\psi_i \hat{b}_{T+1}^i = 0$$

It is straightforward to see that these conditions imply:

$$\beta^T u'_i(\hat{c}_T^i) \hat{b}_{T+1}^i = 0$$

When we extend the number of periods to infinity, we see that this condition is:

$$\lim_{T \rightarrow \infty} \beta^T u'_i(\hat{c}_T^i) \hat{b}_{T+1}^i = 0$$

Which is exactly the TVC from above.

We can also derive the TVC using the optimality conditions from the "Characterizing a SME" section. Begin with the budget constraint at $t = 0$, using the initial condition for the bond $\hat{b}_0^i = 0$.

$$\hat{c}_0^i + \hat{b}_1^i = w_0^i$$

Rearranging the $t = 1$ budget constraint, we get:

$$\hat{b}_1^i = \frac{1}{(1 + \hat{r}_1)} (\hat{c}_1^i - w_1^i + \hat{b}_2^i)$$

Plug this into the $t = 0$ budget constraint:

$$\hat{c}_0^i + \frac{1}{(1 + \hat{r}_1)} (\hat{c}_1^i - w_1^i + \hat{b}_2^i) = w_0^i$$

Using the budget constraint at period t , we can derive the formula for $\hat{b}_t^i$:

$$\hat{b}_t^i = \frac{1}{(1 + \hat{r}_t)} (\hat{c}_t^i - w_t^i + \hat{b}_{t+1}^i)$$

Using this, we can plug in the equation for $\hat{b}_2^i$ into our $t = 0$ budget constraint again:

$$\hat{c}_0^i + \frac{1}{(1 + \hat{r}_1)} \left(\hat{c}_1^i - w_1^i + \frac{1}{(1 + \hat{r}_2)} (\hat{c}_2^i - w_2^i + \hat{b}_3^i) \right) = w_0^i$$

Through repeated substitution until period t , we can derive:

$$\hat{c}_0^i + \frac{\hat{c}_1^i - w_1^i}{(1 + \hat{r}_1)} + \frac{\hat{c}_2^i - w_2^i}{(1 + \hat{r}_1)(1 + \hat{r}_2)} + \dots + \frac{\hat{c}_t^i - w_t^i}{(1 + \hat{r}_1)(1 + \hat{r}_2) \dots (1 + \hat{r}_t)} + \frac{\hat{b}_{t+1}^i}{(1 + \hat{r}_1)(1 + \hat{r}_2) \dots (1 + \hat{r}_t)} = w_0^i$$

Which can be rearranged to:

$$\begin{aligned} \hat{c}_0^i + \frac{\hat{c}_1^i}{(1 + \hat{r}_1)} + \frac{\hat{c}_2^i}{(1 + \hat{r}_1)(1 + \hat{r}_2)} + \dots + \frac{\hat{c}_t^i}{(1 + \hat{r}_1)(1 + \hat{r}_2) \dots (1 + \hat{r}_t)} + \frac{\hat{b}_{t+1}^i}{(1 + \hat{r}_1)(1 + \hat{r}_2) \dots (1 + \hat{r}_t)} \\ = w_0^i + \frac{w_1^i}{(1 + \hat{r}_1)} + \frac{w_2^i}{(1 + \hat{r}_1)(1 + \hat{r}_2)} + \dots + \frac{w_t^i}{(1 + \hat{r}_1)(1 + \hat{r}_2) \dots (1 + \hat{r}_t)} \end{aligned}$$

A necessary condition for this sum to converge is for the bond term to converge to zero as $t \rightarrow \infty$. Thus in any equilibrium, we must require that:

$$\lim_{t \rightarrow \infty} \frac{\hat{b}_{t+1}^i}{(1 + \hat{r}_1)(1 + \hat{r}_2) \dots (1 + \hat{r}_t)} = 0$$

Using the FOCs for bonds, we can see that:

$$(1 + \hat{r}_1)(1 + \hat{r}_2) \dots (1 + \hat{r}_t) = \frac{\mu_0^i}{\mu_1^i} \frac{\mu_1^i}{\mu_2^i} \dots \frac{\mu_{t-1}^i}{\mu_t^i} = \frac{\mu_0^i}{\mu_t^i}$$

Plugging this into the limit, we get the TVC for this problem:

$$\lim_{t \rightarrow \infty} \frac{\hat{b}_{t+1}^i}{\frac{\mu_0^i}{\mu_t^i}} = 0 \iff \lim_{t \rightarrow \infty} \mu_t^i \hat{b}_{t+1}^i = 0 \iff \lim_{t \rightarrow \infty} \beta^t u_i'(\hat{c}_t^i) \hat{b}_{t+1}^i = 0$$