

Macro Theory (8105) Recitation 1

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1 Components of Economic Models

Broadly defined, economics is the study of decision making under constraints.¹ As economists, we want to model these decision processes and the interactions between decision makers. Under this definition, we see that an economic model must describe:

- **Agents:** Who are the decision makers? What are their defining characteristics (e.g. preferences, endowments, technology)?
- **Decisions/Choices:** What kind of choices/decisions can agents make?
- **Constraints:** What constraints must the agents abide by?
- **Information:** What do the agents know?
- **Interactions:** How do the agents interact with one another?

This is the general framework we will use when we write down economic models in this course.

2 Competitive Equilibrium

The last bullet point in the list above, Interactions, is closely related to the idea of an **equilibrium concept**. An equilibrium concept describes the relationship between the agents and market prices. In a **competitive equilibrium**, the equilibrium concept we will focus on, households and firms take prices as given. That is, these agents cannot deliberately affect prices through their actions.

In our exercise today, we will operate within a pure exchange economy (also known as an endowment economy). This means that each household/consumer

¹These notes are based off of Professor Tim Kehoe's lectures, Alex Wurdinger's Fall 2022 recitations, Professor Dirk Krueger's lecture notes, and past Econ 8105 TAs' (Amol Amol, Conor Ryan, Monica Tran Xuan) recitation notes. I do not claim credit for these notes and errors are my own.

has an exogenously given sequence of the consumption good;² there is no production. The households can trade/exchange claims to their endowed consumption goods across time using the price system. Written in words, the definition of a competitive equilibrium in a pure exchange economy is as follows:

A **competitive equilibrium** in a pure exchange economy consists of:

- 1.) Allocations for each household, and
- 2.) Price System, such that
 - (HH) Given prices, each household chooses their allocation to maximize utility subject to their budget constraint.
 - (MC) The market clears (supply=demand) each period.

You will see that our definitions of an **Arrow-Debreu Equilibrium** and **Sequential Markets Equilibrium** within a pure exchange economy follow this general definition.

3 Arrow-Debreu Equilibrium

Now we will describe and define a specific type of competitive equilibrium, called an Arrow-Debreu Equilibrium. To begin, we will describe the environment we will operate within:

- **Time:** $t = 0, 1, 2, 3, \dots$
- **Commodities:** One nonstorable consumption good per period
- **Agents:**
 - Two infinitely lived households $i \in I = \{1, 2\}$
 - Preferences over a sequence of consumption $c^i = (c_0^i, c_1^i, c_2^i, \dots)$ are represented by:

$$U_i(c^i) = \sum_{t=0}^{\infty} \beta_i^t u_i(c_t^i)$$

where $\beta_i \in (0, 1)$ is the discount factor.³

- Each agent has an endowment $\{w_t^i\}_{t=0}^{\infty} = \{w_0^i, w_1^i, w_2^i, \dots\}$, where w_t^i denotes the amount of the consumption good endowed to agent i in period t
- Both agents have perfect information

- **Market Structure:**

²In recitation I called this per-period income, but it is more correct to call this an endowment of goods each period.

³Note that β_i indicates that the discount factor can be different across households. In most of our exercises, however, households will have the same discount factor $\beta_i = \beta \forall i \in I$.

- Market opens at $t = 0$ and agents trade claims to commodities across all dates (future contracts)
- Market closes when agents are finished trading future contracts (still $t = 0$)
- Futures contracts are perfectly enforced

• **Prices:**

- Single price for the commodity in each period: $\{p_t\}_{t=0}^{\infty}$
- p_t is the period zero price of a single unit of consumption in period t . I.e., if an agent buys one unit of consumption good in period t , the seller receives the value of p_t units of period 0 consumption
- Prices are taken as given

We have completed the description of our environment.

Definition: An **Arrow-Debreu Equilibrium (ADE)** consists of:

- 1.) Household Allocations: $\hat{Z}^i \equiv \{\hat{c}_t^i\}_{t=0}^{\infty} \forall i \in \{1, 2\}$
- 2.) Price System: $\hat{P} \equiv \{\hat{p}_t\}_{t=0}^{\infty}$

such that:

(HH) Given $\hat{P}$, household $i \in \{1, 2\}$ chooses $\hat{Z}^i$ to solve:

$$\begin{aligned} & \max_{\{\hat{c}_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u_i(\hat{c}_t^i) \\ s.t. \quad & \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^i \leq \sum_{t=0}^{\infty} \hat{p}_t w_t^i \\ & \hat{c}_t^i \geq 0 \quad \forall t \end{aligned}$$

(MC) Market clears in each period:

$$\sum_{i \in I} \hat{c}_t^i = \sum_{i \in I} w_t^i \quad \forall t$$

This completes the definition. Note that the hatted consumption and prices indicate that these variables are determined in equilibrium.

4 Characterizing an ADE

To characterize an equilibrium, we must write down all of the conditions that are necessary⁴ for the equilibrium solution. We call these conditions the **optimality conditions**.

⁴And potentially sufficient, depending on the problem. For example, in this exercise the optimality conditions are both necessary and sufficient for the equilibrium solution if the utility functions are concave.

We will assume that $u_i(c_t^i)$ is differentiable for each i . We can write the Lagrangian for each agent:

$$\mathcal{L}^i = \sum_{t=0}^{\infty} \beta^t u_i(c_t^i) + \lambda_i \left(\sum_{t=0}^{\infty} \hat{p}_t w_t^i - \sum_{t=0}^{\infty} \hat{p}_t c_t^i \right) + \sum_{t=0}^{\infty} \theta_t^i c_t^i$$

Now, consider the Karush-Kuhn-Tucker (KKT) first order conditions (FOCs):⁵

$$\begin{aligned} \frac{\partial \mathcal{L}^i}{\partial c_t^i} = 0 & \quad : & \quad \beta^t u_i'(\hat{c}_t^i) - \lambda_i \hat{p}_t + \theta_t^i = 0 \\ \frac{\partial \mathcal{L}^i}{\partial \lambda_i} \geq 0 & \quad : & \quad \sum_{t=0}^{\infty} \hat{p}_t w_t^i - \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^i \geq 0 \\ \lambda_i \geq 0 & \quad : & \quad \lambda_i \geq 0 \\ \lambda_i \frac{\partial \mathcal{L}^i}{\partial \lambda_i} = 0 & \quad : & \quad \lambda_i \left(\sum_{t=0}^{\infty} \hat{p}_t w_t^i - \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^i \right) = 0 \\ \frac{\partial \mathcal{L}^i}{\partial \theta_t^i} \geq 0 & \quad : & \quad \hat{c}_t^i \geq 0 \\ \theta_t^i \geq 0 & \quad : & \quad \theta_t^i \geq 0 \\ \theta_t^i \frac{\partial \mathcal{L}^i}{\partial \theta_t^i} = 0 & \quad : & \quad \theta_t^i \hat{c}_t^i = 0 \end{aligned}$$

We can succinctly write these conditions in an equivalent form:⁶

$$\sum_{t=0}^{\infty} \hat{p}_t w_t^i - \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^i \geq 0 \quad (=0 \text{ if } \lambda_i > 0)$$

$$\beta^t u_i'(\hat{c}_t^i) - \lambda_i \hat{p}_t \leq 0 \quad (=0 \text{ if } \hat{c}_t^i > 0)$$

Under what assumptions do these two equations hold with equality? For the budget constraint to bind (the first equation equal to zero), we can assume that our period utility is strictly increasing in consumption. I.e. $u_i(\tilde{c}) > u_i(c)$ if $\tilde{c} > c$. It is intuitive why the budget constraint would bind if the agent is strictly better off by eating more. Such an agent would not be maximizing their utility if they didn't spend all of their wealth!

To guarantee that equilibrium consumption is positive in all periods⁷ (so that

⁵Sundaram's "A First Course in Optimization Theory" chapter 6 is a good place to review the KKT theorem.

⁶Note that the conditions $\lambda_i \frac{\partial \mathcal{L}^i}{\partial \lambda_i} = 0$ and $\theta_t^i \frac{\partial \mathcal{L}^i}{\partial \theta_t^i} = 0$ are called the "complementary slackness" conditions. It tells us that if a constraint isn't binding, its corresponding Lagrange multiplier must be equal to zero.

⁷When $\hat{c}_t^i = 0$ for some period, we say that we have a corner solution. If you think there is a corner solution for period t , you can break the problem into two cases: 1.) $\theta_t^i > 0 \implies \hat{c}_t^i = 0$,

$\beta^t u'_i(\hat{c}_t^i) = \lambda_i \hat{p}_t$), we can assume our period utility function satisfies the first of the two **Inada conditions**:

$$\begin{aligned}\lim_{c \rightarrow 0} u'_i(c) &= +\infty \\ \lim_{c \rightarrow \infty} u'_i(c) &= 0\end{aligned}$$

In layman's terms, the first condition says that even the tiniest bit of consumption increases the agent's utility an infinite amount when they were originally consuming nothing. This essentially means that it will never be optimal for the agent to consume 0.⁸ Therefore, if our period utility function satisfies this condition, we know that $\hat{c}_t^i > 0$ in equilibrium.

Using these assumptions, we can rewrite our KKT FOCs as:

$$\begin{aligned}\beta^t u'_i(\hat{c}_t^i) &= \lambda_i \hat{p}_t \quad (1) \\ \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^i &= \sum_{t=0}^{\infty} \hat{p}_t w_t^i\end{aligned}$$

Using (1) for household i, we can derive the Euler equation:

$$\frac{u'_i(\hat{c}_t^i)}{\beta u'_i(\hat{c}_{t+1}^i)} = \frac{\hat{p}_t}{\hat{p}_{t+1}} \quad \forall t$$

To fully characterize our ADE, we need the consumption first order conditions (FOCs),⁹ the budget constraints, and the market clearing condition.

5 Calculating an ADE

Now that we have characterized the ADE, we will calculate the equilibrium given specific utility functions and endowments. For this exercise, we will assume:

- $u_1(c) = u_2(c) = \ln(c)$
- $\{w_t^1\}_{t=0}^{\infty} = \{a, 0, a, 0, \dots\}$

and 2.) $\theta_t^i = 0$. Then use the optimality conditions to solve for the optimal consumption in each separate case.

⁸The second condition is not relevant for this example, but it means that as consumption tends to infinity, an extra unit of consumption has basically no affect on the agent's utility. These two conditions are useful because they ensure that the optimal consumption in each period will be positive and finite. An example of period utility function that satisfies the Inada conditions and strict increasing is log utility ($u(c) = \ln(c)$).

⁹An earlier version of this document said Euler equation instead of the consumption FOCs. While I think it is true that you can use the Euler condition instead of the FOCs, I will stick with how Professor Kehoe characterizes the equilibrium in his notes.

- $\{w_t^2\}_{t=0}^\infty = \{0, a, 0, a, \dots\}$
- $a > 0$

Our optimality conditions are:

$$\begin{aligned}\beta^t u'_i(\hat{c}_t^i) &= \lambda_i \hat{p}_t \quad \forall t, i \in \{1, 2\} \\ \sum_{t=0}^\infty \hat{p}_t \hat{c}_t^i &= \sum_{t=0}^\infty \hat{p}_t w_t^i \quad \forall i \in \{1, 2\} \\ \hat{c}_t^1 + \hat{c}_t^2 &= w_t^1 + w_t^2 = a \quad \forall i \in \{1, 2\}\end{aligned}$$

Divide the consumption FOC in period t for HH 1 by the FOC in period t of HH 2:

$$\begin{aligned}\frac{\beta^t u'_1(\hat{c}_t^1)}{\beta^t u'_2(\hat{c}_t^2)} &= \frac{\frac{1}{\hat{c}_t^1}}{\frac{1}{\hat{c}_t^2}} = \frac{\hat{c}_t^2}{\hat{c}_t^1} = \frac{\lambda_1}{\lambda_2} \\ \implies \hat{c}_t^2 &= \frac{\lambda_1}{\lambda_2} \hat{c}_t^1\end{aligned}$$

Now, plug this into the market clearing condition.

$$\begin{aligned}\hat{c}_t^1 + \hat{c}_t^2 &= \hat{c}_t^1 + \frac{\lambda_1}{\lambda_2} \hat{c}_t^1 = \hat{c}_t^1 \left(1 + \frac{\lambda_1}{\lambda_2}\right) = w_t^1 + w_t^2 = a \\ \implies \hat{c}_t^1 \left(1 + \frac{\lambda_1}{\lambda_2}\right) &= \hat{c}_t^1 \left(\frac{\lambda_1 + \lambda_2}{\lambda_2}\right) = a \\ \implies \hat{c}_t^1 &= \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right) a\end{aligned}$$

Plug $\hat{c}_t^1$ back into the market clearing condition to get $\hat{c}_t^2$:

$$\begin{aligned}\hat{c}_t^1 + \hat{c}_t^2 &= \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right) a + \hat{c}_t^2 = a \\ \implies \hat{c}_t^2 &= a - \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right) a \\ \implies \hat{c}_t^2 &= \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right) a\end{aligned}$$

We have now solved for $\hat{c}_t^1$ and $\hat{c}_t^2 \forall t$. We see that optimal consumption for both households is constant across time ($\hat{c}_t^i = \hat{c}_{t'}^i = \bar{c}^i \forall t, t', i \in \{1, 2\}$). Both households consume a fixed share of the aggregate endowment in all periods.

Using this information, we can solve for prices. Divide the consumption FOC at time t by the consumption FOC in time zero:

$$\frac{\beta^t u'_i(\hat{c}_t^i)}{\beta^0 u'_i(\hat{c}_0^i)} = \frac{\beta^t \frac{1}{\hat{c}_t^i}}{\frac{1}{\hat{c}_0^i}} = \frac{\beta^t \hat{c}_0^i}{\hat{c}_t^i} = \frac{\hat{p}_t}{\hat{p}_0}$$

$$\implies \beta^t = \frac{\hat{p}_t}{\hat{p}_0}$$

We can normalize the period zero price¹⁰ to equal 1 i.e. $\hat{p}_0 = 1$. Thus we find that $\hat{p}_t = \beta^t \forall t$.¹¹ Why do we normalize the period zero price to 1? *Relative prices* are all that matter to these households. Suppose we had an equilibrium consisting of allocations $\{\{\hat{c}_t^i\}_{i \in \{1,2\}}\}_{t=0}^\infty$ and prices $\{\hat{p}_t\}_{t=0}^\infty$. Now, suppose we multiply all prices by $\mu > 0$. It is straightforward to see that both households' budget constraint remains unchanged and therefore their maximization problems is unchanged:

$$\sum_{t=0}^\infty (\mu \hat{p}_t) \hat{c}_t^i \leq \sum_{t=0}^\infty (\mu \hat{p}_t) w_t^i \iff \mu \sum_{t=0}^\infty \hat{p}_t \hat{c}_t^i \leq \mu \sum_{t=0}^\infty \hat{p}_t w_t^i \iff \sum_{t=0}^\infty \hat{p}_t \hat{c}_t^i \leq \sum_{t=0}^\infty \hat{p}_t w_t^i$$

Since we know that consumption is constant over time ($\hat{c}_t^i = \bar{c}^i = \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right) a \forall t$) and that $\hat{p}_t = \beta^t \forall t$, we can write:

$$\hat{p}_t \hat{c}_t^i = \beta^t \bar{c}^i$$

Plug this into household i's budget constraint (which we know binds because of the strictly increasing utility function):

$$\sum_{t=0}^\infty \hat{p}_t \hat{c}_t^i = \sum_{t=0}^\infty \beta^t \bar{c}^i = \bar{c}^i \sum_{t=0}^\infty \beta^t = \frac{\bar{c}^i}{1 - \beta} = \sum_{t=0}^\infty \hat{p}_t w_t^i = \sum_{t=0}^\infty \beta^t w_t^i$$

Now, let's solve for household 1's consumption $\hat{c}_t^1 = \bar{c}^1 \forall t$. Household 1 receives an endowment a of the consumption good in even periods and an endowment of 0 in odd periods. The right-hand side of their budget constraint is:

$$\sum_{t=0}^\infty \beta^t w_t^i = \sum_{t=0}^\infty \beta^{2t} a + \sum_{t=0}^\infty \beta^{2t+1} 0 = \sum_{t=0}^\infty \beta^{2t} a = \frac{a}{1 - \beta^2}$$

Setting the left-hand side equal to the right hand-side of the budget constraint, we get:

$$\begin{aligned} \frac{\bar{c}^1}{1 - \beta} &= \frac{a}{1 - \beta^2} \\ \implies \bar{c}^1 &= \frac{a(1 - \beta)}{1 - \beta^2} = \frac{a}{1 + \beta} \end{aligned}$$

Plugging this into market clearing, we solve for $\bar{c}^2$:

$$\bar{c}^2 = \frac{a\beta}{1 + \beta}$$

¹⁰We call consumption in period 0 the **numeraire**.

¹¹Since $\beta \in (0, 1)$, we see that prices are decreasing over time and that $\hat{p}_t \rightarrow 0$ as $t \rightarrow \infty$.

We have found the optimal allocations and prices and thus have calculated the (unique) ADE in this economy.

$$\hat{c}_t^1 = \frac{a}{1+\beta} \quad \forall t$$

$$\hat{c}_t^2 = \frac{a\beta}{1+\beta} \quad \forall t$$

$$\hat{p}_t = \beta^t \quad \forall t$$

6 Sequential Markets Equilibrium

We will now define another type of competitive equilibrium, a **Sequential Markets Equilibrium (SME)**. The environment for an SME only differs from an ADE in the market structure and price system.

- **Market Structure:**

- Markets are open each period
- Agents exchange one period bonds with each other

- **Price System:**

- Interest rates on the one-period bonds: $\{r_{t+1}\}_{t=0}^{\infty}$
- If an agent borrows b_{t+1}^i in period t , they must repay $(1+r_{t+1})b_{t+1}^i$ in period $t+1$

We are now prepared to define a SME.

Definition: A **Sequential Markets Equilibrium** consists of:

1.) Household allocations: $\hat{Z}^i \equiv \{\hat{c}_t^i, \hat{b}_{t+1}^i\}_{t=0}^{\infty} \quad \forall i \in \{1, 2\}$

2.) Interest rates: $\hat{P} \equiv \{\hat{r}_{t+1}\}_{t=0}^{\infty}$ such that:

(HH) Given $\hat{P}$, household i chooses $\hat{Z}^i$ to solve:

$$\begin{aligned} & \max_{\{c_t^i, b_{t+1}^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u_i(c_t^i) \\ \text{s.t.} \quad & c_t^i + b_{t+1}^i \leq (1 + \hat{r}_t)b_t^i + w_t^i \quad \forall t \\ & b_{t+1}^i \geq -B \quad \forall t \\ & b_0^i = \bar{b}_i = 0 \quad \text{given} \\ & c_t^i \geq 0 \quad \forall t \end{aligned}$$

(MC) The goods market and bond market clears in each period:

$$\sum_{i \in I} \hat{c}_t^i = \sum_{i \in I} w_t^i \quad \forall t$$

$$\sum_{i \in I} \hat{b}_{t+1}^i = 0 \quad \forall t$$

This completes our definition of an SME.

A key component of the SME is the no Ponzi condition $b_{t+1}^i \geq -B$. As Professor Kehoe mentioned in lecture, this constraint prevents the agents from running a Ponzi scheme.¹²

We will not characterize the SME in these notes, but you should try to write down the Lagrangian as practice. Note that we will also need a **transversality condition** to fully characterize the equilibrium, but we will talk about that later.

A very neat feature about ADE and SME is that their equilibrium consumption allocations are identical. In this sense, ADE and SME are equivalent. You will prove this equivalence theorem in the problem sets.

¹²As a technical note, in this model we want B to be large enough so that this constraint never binds.